The Mean Field Ensemble Kalman Filter: Near-Gaussian Setting

Journées MAS 2024

Urbain Vaes
urbain.vaes@inria.fr

MATHERIALS – Inria Paris & CERMICS – École des Ponts ParisTech

August 30, 2024

Collaborators

José Carrillo

Mathematical Institute

Franca Hoffmann

Hausdorff Center for Mathematics

Andrew Stuart

Department of Computing + Mathematical Sciences

References:

- The Mean Field Ensemble Kalman Filter: Near-Gaussian Setting, arXiv preprint, 2022;
- Statistical Accuracy of Approximate Filtering Methods, arXiv preprint, 2024.

Outline

The discrete-time filtering problem

Error estimate for the mean field ensemble Kalman filter

Conclusions and perspectives

Notation: Probability measures and densities, Operators

The discrete-time filtering problem

State dynamics and observations

$$
\begin{aligned}
\textbf{Stochastic dynamics:} \quad & v_{n+1} = \Psi(v_n) + \xi_n, && \xi_n \sim \mathsf{N}(0, \Sigma), \\
\textbf{Data model:} \quad & y_{n+1} = h(v_{n+1}) + \eta_{n+1}, && \eta_{n+1} \sim \mathsf{N}(0, \Gamma).
\end{aligned}
$$

Independence assumption:

$$v_0 \perp\!\!\!\perp \{\xi_n\} \perp\!\!\!\perp \{\eta_n\}$$

Initial state: $v_0 \sim \mathsf{N}(m_0, C_0)$.

Notations:

- $\{v_n\}_{n \in [\![0,N]\!]}$ is the unknown state in $\mathbf{R}^d$.
- $\{y_n\}_{n \in [\![1,N]\!]}$ are the observations in $\mathbf{R}^K$.
- $\Psi \colon \mathbf{R}^d \to \mathbf{R}^d$ and $h \colon \mathbf{R}^d \to \mathbf{R}^K$ are nonlinear operators.
- $Y_n^\dagger = \{y_1^\dagger, \dots, y_n^\dagger\}$ is a given realization of the data up to time n.

Goal: Approximate sequentially $\mu_n = \mathrm{Law}(v_n | Y_n^\dagger)$.

Key Linear Operators on $\mathcal{P}$

Definition of $\mathcal{P}$, $\mathcal{G}$

- $\mathcal{P}(\mathbf{R}^r)$: all probability measures on $\mathbf{R}^r$.
- $\mathcal{G}(\mathbf{R}^r)$: all Gaussian probability measures on $\mathbf{R}^r$.

Definition of P

P: $\mathcal{P}(\mathbf{R}^d) \to \mathcal{P}(\mathbf{R}^d)$ is the linear operator:

$$\mathsf{P}\pi(u) = \frac{1}{\sqrt{(2\pi)^d \det \Sigma}} \int \exp\left(-\frac{1}{2}|u - \Psi(v)|_\Sigma^2\right) \pi(v)\, \mathrm{d}v.$$

Definition of Q

Q: $\mathcal{P}(\mathbf{R}^d) \to \mathcal{P}(\mathbf{R}^d \times \mathbf{R}^K)$ is the linear operator:

$$\mathsf{Q}\pi(u, y) = \frac{1}{\sqrt{(2\pi)^K \det \Gamma}} \exp\left(-\frac{1}{2}\big|y - h(u)\big|_\Gamma^2\right) \pi(u).$$

Unconditioned Dynamics

State dynamics and observations

$$\begin{aligned} \textbf{Stochastic dynamics:} \quad & v_{n+1} = \Psi(v_n) + \xi_n, \qquad && \xi_n \sim \mathsf{N}(0, \Sigma), \\ \textbf{Data model:} \quad & y_{n+1} = h(v_{n+1}) + \eta_{n+1}, \qquad && \eta_{n+1} \sim \mathsf{N}(0, \Gamma). \end{aligned}$$

Independence assumption:

$$v_0 \perp\!\!\!\perp \{\xi_n\} \perp\!\!\!\perp \{\eta_n\}$$

Initial state: $v_0 \sim \mathsf{N}(m_0, C_0)$.

Evolution of unconditioned dynamics

Let $v_n \sim \pi_n$ and $(v_n, y_n) \sim \chi_n$. Then

$$\begin{aligned} \pi_{n+1} &= \mathsf{P}\pi_n, \\ \chi_{n+1} &= \mathsf{Q}\pi_{n+1} \end{aligned}$$

Evolution of the true filtering distribution

Key Nonlinear Operator on $\mathcal{P}$: conditioning

$\mathsf{B}(\,\bullet\,; y^\dagger)\colon \mathcal{P}(\mathbf{R}^d \times \mathbf{R}^K) \to \mathcal{P}(\mathbf{R}^d)$ describes conditioning on observation $y = y^\dagger$:

$$\mathsf{B}(\rho; y^\dagger)(u) = \frac{\rho(u, y^\dagger)}{\int_{\mathbf{R}^d} \rho(U, y^\dagger)\,\mathrm{d}U}.$$

Probability Viewpoint (Nonlinear)

Notation: $Y_n^\dagger = \{y_\ell^\dagger\}_{\ell=1}^n, \quad v_n | Y_n^\dagger \sim \mu_n.$

$$\begin{aligned}
\widehat{\mu}_{n+1} &= \mathsf{P}\mu_n, & v_{n+1}|Y_n^\dagger &\sim \widehat{\mu}_{n+1}\\
\rho_{n+1} &= \mathsf{Q}\widehat{\mu}_{n+1}, & (v_{n+1}, y_{n+1})|Y_n^\dagger &\sim \rho_{n+1}\\
\mu_{n+1} &= \mathsf{B}(\rho_{n+1}; y_{n+1}^\dagger), & &\text{conditioning.}
\end{aligned}$$

Schematically,

$$\mathrm{Law}(v_n|Y_n^\dagger) \xrightarrow{\mathsf{P}} \mathrm{Law}(v_{n+1}|Y_n^\dagger) \xrightarrow{\mathsf{B}\circ\mathsf{Q}} \mathrm{Law}(v_{n+1}|Y_{n+1}^\dagger).$$

The True Filter

Sequential Interleaving of Prediction and Bayes Theorem

$\mathsf{P}\mu_n$ is prior prediction; $\mathsf{L}(\bullet; y^\dagger) := \mathsf{B}(\bullet; y^\dagger) \circ \mathsf{Q}$ maps prior to posterior:

$$\mu_{n+1} = \mathsf{B}(\mathsf{Q}\mathsf{P}\mu_n; y^\dagger_{n+1}),$$
$$\mu_{n+1} = \mathsf{L}(\mathsf{P}\mu_n; y^\dagger_{n+1}).$$

Particle Filter[1]

$\mathsf{S}^J : \mathcal{P}(\mathbf{R}^r) \times \Omega \to \mathcal{P}(\mathbf{R}^r)$ is empirical approximation operator:

$$\mathsf{S}^J \mu = \frac{1}{J}\sum_{j=1}^{J} \delta_{v_j}, \qquad v_j \overset{\text{i.i.d.}}{\sim} \mu\,.$$

S^J is thus a random approximation of the identity operator on $\mathcal{P}(\mathbf{R}^r)$.

$$\mu^{\mathrm{PF}}_{n+1} = \mathsf{L}(\mathsf{S}^J\mathsf{P}\mu^{\mathrm{PF}}_n; y^\dagger_{n+1}).$$

[1] A. Doucet, N. de Freitas, and N. Gordon, editors. Statistics for Engineering and Information Science. Springer-Verlag, New York, 2001.

Convergence of the Particle Filter (1/2)

Convergence of the particle filter[2][3]

$$\sup_{0 \leqslant n \leqslant N} d\left(\mu_n, \mu_n^{\mathrm{PF}}\right) \leqslant \frac{C}{\sqrt{J}}, \qquad d(\mu, \nu)^2 := \sup_{|f| \leqslant 1} \mathbf{E}\big(\mu[f] - \nu[f]\big)^2,$$

Comments on proof[4][5]

- Metric $d(\bullet, \bullet)$ on random probability measures:
- Reduces to TV between deterministic measures.
- Consistency + Stability Implies Convergence.
- Consistency: $d(\mathsf{S}^J \mu, \mu) \leqslant \frac{1}{\sqrt{J}}$.
- Stability: P, L Lipschitz in $d(\bullet, \bullet)$.
- Suffers from weight collapse.

[2] P. Del Moral. C. R. Acad. Sci. Paris Sér. I Math., 1997.
[3] P. Del Moral and A. Guionnet. Ann. Inst. H. Poincaré Probab. Statist., 2001.
[4] P. Rebeschini and R. van Handel. Ann. Appl. Probab., 2015.
[5] D. Sanz-Alonso, A. Stuart, and A. Taeb. Cambridge University Press, 2023.

Convergence of the Particle Filter (2/2)

True filtering distribution and particle filter:

$$\mu_{n+1} = \mathsf{L}(\mathsf{P}\mu_n; y^\dagger_{n+1})$$
$$\mu^{\mathrm{PF}}_{n+1} = \mathsf{L}(\mathsf{S}^J \mathsf{P}\mu^{\mathrm{PF}}_n; y^\dagger_{n+1}).$$

Sketch of proof

Consistency. Monte Carlo error

$$\forall \mu \in \mathcal{P}(\mathbf{R}^d), \qquad d(\mathsf{S}^J\mu, \mu) \leqslant \frac{1}{\sqrt{J}}.$$

Stability. Under appropriate assumptions,

$$\forall (\mu,\nu) \in \mathcal{P}(\mathbf{R}^d)\times\mathcal{P}(\mathbf{R}^d), \qquad d(\mathsf{P}\mu, \mathsf{P}\nu) \leqslant d(\mu,\nu), \qquad d(\mathsf{L}\mu, \mathsf{L}\nu) \leqslant \ell_L d(\mu,\nu).$$

Main argument.

$$\begin{aligned} d(\mu_{n+1}, \mu^{\mathrm{PF}}_{n+1}) &\leqslant \ell_L d\left(\mathsf{P}\mu_n, \mathsf{S}^J\mathsf{P}\mu^{\mathrm{PF}}_n\right) \\ &\leqslant \ell_L d\left(\mathsf{P}\mu_n, \mathsf{P}\mu^{\mathrm{PF}}_n\right) + \ell_L d\left(\mathsf{P}\mu^{\mathrm{PF}}_n, \mathsf{S}^J\mathsf{P}\mu^{\mathrm{PF}}_n\right) \\ &\leqslant \ell_L d\left(\mu_n, \mu^{\mathrm{PF}}_n\right) + \frac{\ell_L}{\sqrt{J}}. \end{aligned}$$

Weights

Particle Filter (Weight Collapse)

$$
\begin{aligned}
\widehat{v}_{n+1}^{(j)} &= \Psi\big(v_n^{(j)}\big) + \xi_n^{(j)}, \quad v_n^{(j)} \sim \mu_n^{\mathrm{PF}},\\
\ell_{n+1}^{(j)} &= \exp\left(-\frac{1}{2}\big|y_{n+1}^{\dagger} - h\big(\widehat{v}_{n+1}^{(j)}\big)\big|_{\Gamma}^{2}\right),\\
\mu_{n+1}^{\mathrm{PF}} &= \sum_{j=1}^{J} w_{n+1}^{(j)} \delta_{\widehat{v}_{n+1}^{(j)}}, \quad w_{n+1}^{(j)} = \ell_{n+1}^{(j)} \Big/ \Big(\sum_{m=1}^{J} \ell_{n+1}^{(m)}\Big).
\end{aligned}
$$

Ensemble Kalman Filter (No Weight Collapse!)

$$
\begin{aligned}
\widehat{v}_{n+1}^{(j)} &= \Psi\big(v_n^{(j)}\big) + \xi_n^{(j)},\\
\widehat{y}_{n+1}^{(j)} &= h(\widehat{v}_{n+1}^{(j)}) + \eta_{n+1}^{(j)},\\
v_{n+1}^{(j)} &= \widehat{v}_{n+1}^{(j)} + \mathcal{C}^{vy}\big(\rho_{n+1}^{\mathrm{EK},J}\big)\mathcal{C}^{yy}\big(\rho_{n+1}^{\mathrm{EK},J}\big)^{-1}\big(y_{n+1}^{\dagger} - \widehat{y}_{n+1}^{(j)}\big),\\
\rho_{n+1}^{\mathrm{EK},J} &= \frac{1}{J}\sum_{j=1}^{J} \delta_{\big(\widehat{v}_{n+1}^{(j)}, \widehat{y}_{n+1}^{(j)}\big)}.
\end{aligned}
$$

Outline

The discrete-time filtering problem

Error estimate for the mean field ensemble Kalman filter

Conclusions and perspectives

The ensemble Kalman filter from a mean field perspective[6] (1/2)

Block Form Of State-Data Covariance

Write covariance under $\rho \in \mathcal{P}(\mathbf{R}^d \times \mathbf{R}^K)$ as:

$$\mathrm{cov}_\rho = \begin{pmatrix} \mathcal{C}^{vv}(\rho) & \mathcal{C}^{vy}(\rho) \\ \mathcal{C}^{vy}(\rho)^\top & \mathcal{C}^{yy}(\rho) \end{pmatrix}.$$

Mean field ensemble Kalman filter

$$\begin{aligned}
\widehat{v}_{n+1} &= \Psi(v_n) + \xi_n, & \xi_n &\sim \mathsf{N}(0, \Sigma), \\
\widehat{y}_{n+1} &= h(\widehat{v}_{n+1}) + \eta_{n+1}, & \eta_{n+1} &\sim \mathsf{N}(0, \Gamma). \\
v_{n+1} &= \widehat{v}_{n+1} + \mathcal{C}^{vy}\big(\rho^{\mathrm{EK}}_{n+1}\big)\mathcal{C}^{yy}\big(\rho^{\mathrm{EK}}_{n+1}\big)^{-1}\big(y^\dagger_{n+1} - \widehat{y}_{n+1}\big), \\
(\widehat{v}_{n+1}, \widehat{y}_{n+1}) &\sim \rho^{\mathrm{EK}}_{n+1}.
\end{aligned}$$

Approximate filtering distribution $\mu^{\mathrm{EK}}_n = \mathrm{Law}(v_n)$.

[6] E. Calvello, S. Reich, and A. M. Stuart. Acta Numerica, 2025.

Evolution equation for the approximate filtering distribution

Key Nonlinear Operator on $\mathcal{P}$

$\mathsf{T}(\bullet; y^\dagger) \colon \mathcal{P}(\mathbf{R}^d \times \mathbf{R}^K) \to \mathcal{P}(\mathbf{R}^d)$ approximates conditioning of ρ on $y = y^\dagger$:

$$
\begin{aligned}
\mathfrak{T}(\bullet, \bullet; \rho, y^\dagger) \colon \mathbf{R}^d \times \mathbf{R}^K &\to \mathbf{R}^d; \\
(v, y) &\mapsto v + \mathcal{C}^{vy}(\rho) \mathcal{C}^{yy}(\rho)^{-1} \bigl(y^\dagger - y\bigr), \\
\mathsf{T}(\rho; y^\dagger) &= \bigl(\mathfrak{T}(\bullet, \bullet; \rho, y^\dagger)\bigr)_\sharp \rho.
\end{aligned}
$$

With this notation:

$$
\begin{aligned}
\mu_{n+1} &= \mathsf{B}(\mathsf{QP}\mu_n; y^\dagger_{n+1}) \\
\mu^{\mathrm{EK}}_{n+1} &= \mathsf{T}(\mathsf{QP}\mu^{\mathrm{EK}}_n; y^\dagger_{n+1}), \qquad \mu^{\mathrm{EK}}_0 = \mu_0.
\end{aligned}
$$

Key fact: $\mathsf{T}(\bullet; y^\dagger) \equiv \mathsf{B}(\bullet; y^\dagger)$ for Gaussian inputs.

$\rightsquigarrow$ mean field ensemble Kalman is exact in the Gaussian setting.

Best Gaussian Approximation in KL

$$\mathsf{G} : \mathcal{P} \to \mathcal{G},$$
$$\mathsf{G}\pi = \operatorname{argmin}_{\mathfrak{p}\in\mathcal{G}}\ d_{\mathrm{KL}}(\pi\|\mathfrak{p}).$$

More concretely, $\mathsf{G}\pi = \mathsf{N}(\mathrm{mean}_\pi, \mathrm{cov}_\pi)$.

Weighted TV Metric

Let $g(v) = 1 + |v|^2$.

$$d_g(\mu_1, \mu_2) = \sup_{|f|\leqslant g} \Big|\mu_1[f] - \mu_2[f]\Big|, \quad \mu[f] = \int f(u)\mu(du).$$

Closeness of exact filter and EnKF (1/2)

Definition

Measure of how close true filter $\{\mu_n\}$ is to being Gaussian:

$$\varepsilon := \sup_{0 \leqslant n \leqslant N} d_g(\mathsf{GQP}\mu_n, \mathsf{QP}\mu_n).$$

Theorem[7]

Let $\mu_0^{\mathrm{EK}} = \mu_0$ and assume that $\|\Psi\|_{L^\infty}, \|h\|_{L^\infty}$ and $|h|_{C^{0,1}}$ are bounded by r. Then there is $C := C(N, r) > 0$ such that

$$\sup_{0 \leqslant n \leqslant N} d_g\left(\mu_n, \mu_n^{\mathrm{EK}}\right) \leqslant C\varepsilon.$$

[7] J. A. Carrillo, F. Hoffmann, A. M. Stuart, and U. Vaes. arXiv preprint, 2022.

Closeness of exact filter and EnKF (2/2)

Assumptions

- Data $Y_j^\dagger$ lies in set

$$B_y := \left\{ Y^\dagger \in \mathbf{R}^{KJ} : \max_{0 \leqslant j \leqslant J} |y_j^\dagger| \leqslant \kappa_y \right\}.$$

- $\Psi_0 \colon \mathbf{R}^d \to \mathbf{R}^d$ and $h_0 \colon \mathbf{R}^d \to \mathbf{R}^K$ are constant functions.

Denote by $B_{\Psi,h}(r)$ the set (Ψ, h) satisfying $\Psi \in B_{L^\infty}(\Psi_0, r)$, $h \in B_{L^\infty}(h_0, r)$.

Corollary[8]

Suppose that the assumptions of the previous theorem and the assumption above are satisfied. Then for any $\epsilon > 0$ there is $\delta > 0$ such that

$$\sup_{Y^\dagger \in B_y} \sup_{(\Psi,h) \in B_{\Psi,h}(\delta)} \left(\sup_{0 \leqslant n \leqslant N} d_g(\mu_n, \mu_n^{\mathrm{EK}}) \right) \leqslant \epsilon.$$

---

[8] J. A. Carrillo, F. Hoffmann, A. M. Stuart, and U. Vaes. arXiv preprint, 2022.

Linear Maps P, Q

The maps P, Q are globally Lipschitz on $\mathcal{P}(\mathbf{R}^d)$ in d_g.

Nonlinear Conditioning Map $\mathsf{B}^{y^\dagger}$

The maps $\mathsf{B}^{y^\dagger}(\bullet) := \mathsf{B}(\bullet; y^\dagger)$ satisfy:

$$\forall \mu \in \mathcal{P}(\mathbf{R}^d) \quad d_g\big(\mathsf{B}^{y^\dagger}(\mathsf{GQP}\mu), \mathsf{B}^{y^\dagger}(\mathsf{QP}\mu)\big) \leqslant \ell_B \, d_g(\mathsf{GQP}\mu, \mathsf{QP}\mu)\,.$$

Let $\mathcal{P}_R$ denote the following subset of probability measures

$$\mathcal{P}_R(\mathbf{R}^r) = \left\{\mu \in \mathcal{P}(\mathbf{R}^r) : \max\left\{|\mathrm{mean}(\mu)|, |\mathrm{cov}(\mu)|^{\frac{1}{2}}, |\mathrm{cov}(\mu)|^{-\frac{1}{2}}\right\} \leqslant R\right\}.$$

Using linearity of $\mathfrak{T}$, which defines nonlinear map $\mathsf{T}^{y^\dagger}$:

Approximate Nonlinear Conditioning Map $\mathsf{T}^{y^\dagger}$

The maps $\mathsf{T}^{y^\dagger}(\bullet) := \mathsf{T}(\bullet; y^\dagger)$ satisfy, using Ψ bounded,

$$\forall(\mu, \rho) \in \mathcal{P}(\mathbf{R}^d) \times \mathcal{P}_R(\mathbf{R}^d \times \mathbf{R}^K),$$
$$d_g\big(\mathsf{T}^{y^\dagger}(\mathsf{QP}\mu), \mathsf{T}^{y^\dagger}(\rho)\big) \leqslant \ell_T(R)\, d_g(\mathsf{QP}\mu, \rho),$$

Moment bounds

Assume that $\|\Psi\|_{L^\infty}, \|h\|_{L^\infty}$ and $\Sigma, \Gamma \succ 0$. Then there is R such that

$$\mathrm{Im}(\mathsf{QP}) \in \mathcal{P}_R(\mathbf{R}^{d+K})$$

Proof of Theorem

Strategy: Consistency + Stability Implies Convergence

Since $\mathsf{T}^{y^\dagger_{n+1}}(\mathsf{G}\bullet) = \mathsf{B}^{y^\dagger_{n+1}}(\mathsf{G}\bullet)$ we have

$$
\begin{aligned}
d_g(\mu^{\mathrm{EK}}_{n+1}, \mu_{n+1}) = d_g&\left(\mathsf{T}^{y^\dagger_{n+1}}(\mathsf{QP}\mu^{\mathrm{EK}}_n), \mathsf{B}^{y^\dagger_{n+1}}(\mathsf{QP}\mu_n)\right) \\
\leqslant d_g&\left(\mathsf{T}^{y^\dagger_{n+1}}(\mathsf{QP}\mu^{\mathrm{EK}}_n), \mathsf{T}^{y^\dagger_{n+1}}(\mathsf{QP}\mu_n)\right) \\
&+ d_g\left(\mathsf{T}^{y^\dagger_{n+1}}(\mathsf{QP}\mu_n), \mathsf{T}^{y^\dagger_{n+1}}(\mathsf{GQP}\mu_n)\right) \\
&\quad + d_g\left(\mathsf{T}^{y^\dagger_{n+1}}(\mathsf{GQP}\mu_n), \mathsf{B}^{y^\dagger_{n+1}}(\mathsf{QP}\mu_n)\right) \\
\leqslant \ell_T(R)\, d_g&\left(\mathsf{QP}\mu^{\mathrm{EK}}_n, \mathsf{QP}\mu_n\right) \\
&+ \ell_T(R)\, d_g\left(\mathsf{QP}\mu_n, \mathsf{GQP}\mu_n)\right) \\
&\quad + d_g\left(\mathsf{B}^{y^\dagger_{n+1}}(\mathsf{GQP}\mu_n), \mathsf{B}^{y^\dagger_{n+1}}(\mathsf{QP}\mu_n)\right) \\
\leqslant c d_g&(\mu^{\mathrm{EK}}_n, \mu_n) + (\ell_T(R) + \ell_B)\,\varepsilon.
\end{aligned}
$$

Outline

The discrete-time filtering problem

Error estimate for the mean field ensemble Kalman filter

Conclusions and perspectives

Perspectives for future work

- Control growth of error with N;
- Extend to continuous-time setting;
- Extend to unbounded setting (in progress);
- Extend to particle approximations[9];
- Relax assumptions of non-zero noises;
- Extend to other transport maps[10].

Thank you for your attention!

[9] F. Le Gland, V. Monbet, and V.-D. Tran. In The Oxford handbook of nonlinear filtering. Oxford Univ. Press, Oxford, 2011.

[10] E. Calvello, S. Reich, and A. M. Stuart. Acta Numerica, 2025.

- Control growth of error with N;
- Extend to continuous-time setting;
- Extend to unbounded setting (in progress);
- Extend to particle approximations[9];
- Relax assumptions of non-zero noises;
- Extend to other transport maps[10].

Thank you for your attention!

[9] F. Le Gland, V. Monbet, and V.-D. Tran. In The Oxford handbook of nonlinear filtering. Oxford Univ. Press, Oxford, 2011.

[10] E. Calvello, S. Reich, and A. M. Stuart. Acta Numerica, 2025.

Some references

E. Calvello, S. Reich, and A. M. Stuart. Ensemble Kalman Methods: A Mean Field Perspective. Acta Numerica, 2025.

J. A. Carrillo, F. Hoffmann, A. M. Stuart, and U. Vaes. The Mean Field Ensemble Kalman Filter: Near-Gaussian Setting. arXiv preprint, 2212.13239, 2022.

P. Del Moral. Nonlinear filtering: interacting particle resolution. C. R. Acad. Sci. Paris Sér. I Math., 325(6):653–658, 1997.

P. Del Moral and A. Guionnet. On the stability of interacting processes with applications to filtering and genetic algorithms. Ann. Inst. H. Poincaré Probab. Statist., 37(2):155–194, 2001.

A. Doucet, N. de Freitas, and N. Gordon, editors. Sequential Monte Carlo methods in practice. Statistics for Engineering and Information Science. Springer-Verlag, New York, 2001.

F. Le Gland, V. Monbet, and V.-D. Tran. Large sample asymptotics for the ensemble Kalman filter. In The Oxford handbook of nonlinear filtering, pages 598–631. Oxford Univ. Press, Oxford, 2011.

P. Rebeschini and R. van Handel. Can local particle filters beat the curse of dimensionality? Ann. Appl. Probab., 25(5):2809–2866, 2015.

D. Sanz-Alonso, A. Stuart, and A. Taeb. Inverse Problems and Data Assimilation. Volume 107. Cambridge University Press, 2023.