Uniform-in-time Mean-field Limit for Consensus-Based Optimization

Groupe de travail marnais: Algos sto et co

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References:

- N. J. Gerber, F. Hoffmann, and UV. ESAIM Control Optim. Calc. Var., 2025
  Mean-field limits for Consensus-Based Optimization and Sampling
- N. J. Gerber, F. Hoffmann, D. Kim, and UV. Arxiv preprint, 2025
  Uniform-in-time propagation of chaos for Consensus-Based Optimization

Outline

Motivating example: Bayesian inverse problems

Paradigmatic inverse problem

Find an unknown parameter $\theta \in \mathcal{U}$ from data $y \in \mathbf{R}^m$ where

$$y = \mathcal{G}(\theta) + \eta,$$

- $\mathcal{G}$ is the forward operator;
- η is observational noise.

Two difficulties[1] associated with this problem are the following:

- Because of the noise, it might be that $y \notin \mathrm{Ran}(\mathcal{G})$;
- The problem might be underdetermined.

Additionally, in many PDE applications,

- $\mathcal{G}$ is expensive to evaluate;
- The derivatives of $\mathcal{G}$ are difficult to calculate;
- θ is a function $\rightarrow$ infinite dimension.

[1] M. Dashti and A. M. Stuart. In Handbook of uncertainty quantification. Vol. 1, 2, 3. Springer, Cham, 2017.

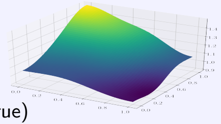
Example: inference of the thermal conductivity in a plate

Mathematical model:

$$
\begin{aligned}
-\nabla \cdot \big(\theta(x)\nabla T(x)\big) &= f(x), && x \in \Omega, \\
T(x) &= 0, && x \in \partial\Omega.
\end{aligned}
$$

Unknown parameter:

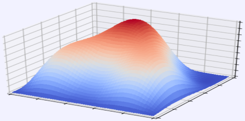
Thermal conductivity $\theta(x)$



(true)

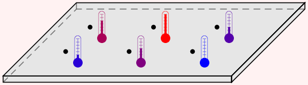
Forward problem

Solution:



Temperature field $T(x)$

Data:

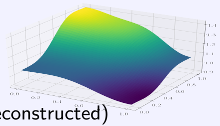


Noisy temperature measurements:

$$y = \big(T(x_1), \dots, T(x_m)\big) + \eta.$$

Inverse problem

MAP estimator:



(reconstructed)

Probabilistic approach for solving "$y = \mathcal{G}(\theta) + \eta$"[1]

Bayesian approach to inverse problems

Modeling step:

- Probability distribution on parameter: $\theta \sim \pi$, encoding our **prior knowledge**;
- Probability distribution for noise: $\eta \sim \nu$.

An application of Bayes' theorem gives the **posterior distribution**:

$$\rho^y(\theta) \propto \pi(\theta)\, \nu\big(y - \mathcal{G}(\theta)\big) = \text{prior} \times \text{likelihood}.$$

(In infinite dimension, use Radon–Nikodym derivative.)

In the Gaussian case where $\pi = \mathcal{N}(m, \Sigma)$ and $\nu = \mathcal{N}(0, \Gamma)$,

$$\rho^y(\theta) \propto \exp\left(-\left(\frac{1}{2}\left|y - \mathcal{G}(\theta)\right|_\Gamma^2 + \frac{1}{2}\left|\theta - m\right|_\Sigma^2\right)\right) =: \exp\big(-\mathcal{F}(\theta)\big).$$

where $|x|_A := \sqrt{x^T A^{-1} x}$.

Two approaches for extracting information:

- Find the maximizer of $\rho^y(\theta)$ (maximum a posteriori estimation);
- Sample the posterior distribution $\rho^y(\theta)$.

[1] A. M. Stuart. Acta Numer., 2010.

Brief review of the recent literature on interacting particle methods

- 1994: Ensemble Kalman filter[1] (6,611 citations);
- 1995: Particle swarm optimization[2] (**90,668** citations);
- 2006: Sequential Monte Carlo samplers[3] (2,255 citations);
- 2010: Affine-invariant many-particle MCMC[4] (3,505 citations);
- 2013: Ensemble Kalman inversion[5] (473 citations);
- 2016: Stein variational gradient descent[6] (1,285 citations);
- 2017: Consensus-based optimization[7] (185 citations);
- 2020: Ensemble Kalman sampling[8] (233 citations);

Often parallelizable, and some can be studied through mean-field equations.

[1] G. Evensen. Journal of Geophysical Research: Oceans, 1994.
[2] J. Kennedy and R. Eberhart. In Proceedings of ICNN'95-international conference on neural networks. ieee, 1995.
[3] P. Del Moral, A. Doucet, and A. Jasra. J. R. Stat. Soc. Ser. B Stat. Methodol., 2006.
[4] J. Goodman and J. Weare. Commun. Appl. Math. Comput. Sci., 2010.
[5] M. A. Iglesias, K. J. H. Law, and A. M. Stuart. Inverse Problems, 2013.
[6] Q. Liu and D. Wang. In Advances In Neural Information Processing Systems, 2016.
[7] R. Pinnau, C. Totzeck, O. Tse, and S. Martin. Math. Models Methods Appl. Sci., 2017.
[8] A. Garbuno-Inigo, F. Hoffmann, W. Li, and A. M. Stuart. SIAM J. Appl. Dyn. Syst., 2020.

Consensus-based optimization (CBO)[1,2]

Global optimization problem:

$$\text{Find } x \in \underset{\mathbf{R}^d}{\arg\min}\, \mathcal{F} \qquad (\mathcal{F} \colon \mathbf{R}^d \to \mathbf{R})$$

CBO interacting particle system

$$\mathrm{d}X_t^j = -\Big(X_t^j - \mathcal{M}_\beta\big(\mu_t^J\big)\Big)\,\mathrm{d}t + \sqrt{2}\sigma\Big|X_t^j - \mathcal{M}_\beta\big(\mu_t^J\big)\Big|\,\mathrm{d}W_t^j, \qquad j = 1, \dots, J,$$

- β is "inverse temperature" parameter.
- μ_t^J is empirical measure $\mu_t^J = \frac{1}{J}\sum_{j=1}^J \delta_{X_t^j}$.
- $\mathcal{M}_\beta \colon \mathcal{P}(\mathbf{R}^d) \to \mathbf{R}^d$ is weighted mean operator:

$$\mathcal{M}_\beta(\mu) = \frac{\int x\, \mathrm{e}^{-\beta\mathcal{F}(x)}\, \mu(\mathrm{d}x)}{\int \mathrm{e}^{-\beta\mathcal{F}(x)}\, \mu(\mathrm{d}x)}, \qquad \mathcal{M}_\beta\big(\mu_t^J\big) = \frac{\sum_{j=1}^J X_t^j \exp\big(-\beta\mathcal{F}(X_t^j)\big)}{\sum_{j=1}^J \exp\big(-\beta\mathcal{F}(X_t^j)\big)}.$$

[1] R. Pinnau, C. Totzeck, O. Tse, and S. Martin. Math. Models Methods Appl. Sci., 2017.
[2] J. A. Carrillo, Y.-P. Choi, C. Totzeck, and O. Tse. Mathematical Models and Methods in Applied Sciences, 2018.

Typical evolution of CBO dynamics

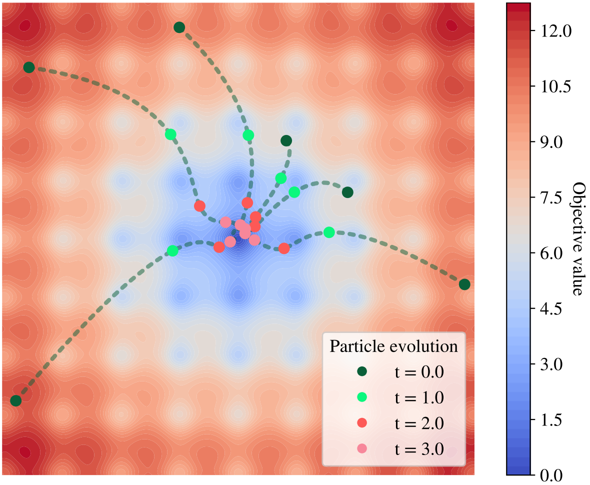
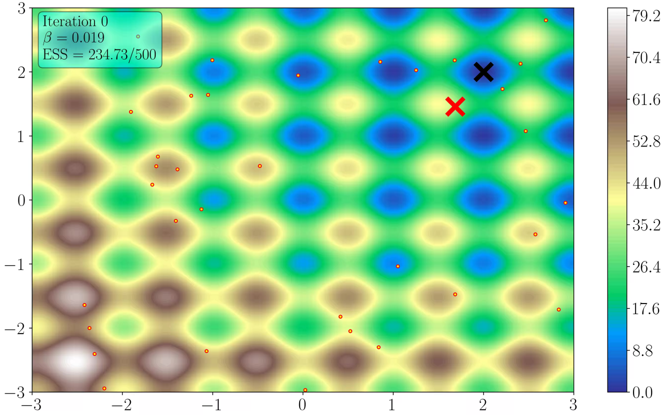


Figure: Ensemble evolution for the Ackley function

Video illustration



Numerical implementation[1]

CBXPy CBX.jl

Software library in Python (lead T. Roith) and Julia (lead R. Bailo):

- Offers high-performance implementation of the method;
- Implements a number of extensions (different noises, mini-batching, sampling, . . .)
- Provides general interface that can accommodate extensions.

JOSS
The Journal of Open Source Software

CBX: Python and Julia Packages for Consensus-Based Interacting Particle Methods

Rafael Bailo [1], Alethea Barbaro [2], Susana N. Gomes [3], Konstantin Riedl [4,5], Tim Roith [6], Claudia Totzeck [7], and Urbain Vaes [8,9]

1 Mathematical Institute, University of Oxford, United Kingdom **2** Delft University of Technology, The Netherlands **3** Mathematics Institute, University of Warwick, United Kingdom **4** Technical University of Munich, Germany **5** Munich Center for Machine Learning, Germany **6** Helmholtz Imaging, Deutsches Elektronen-Synchrotron DESY, Notkestr. 85, 22607 Hamburg, Germany **7** University of Wuppertal, Germany **8** MATHERIALS team, Inria Paris, France **9** École des Ponts ParisTech, Marne-la-Vallée, France

[1] R. Bailo, A. Barbaro, S. N. Gomes, K. Riedl, T. Roith, C. Totzeck, and U. Vaes. Journal of Open Source Software, 2024.

Consensus-based sampling (CBS)[1]

Sampling problem:

$$\text{Generate samples from distribution } \pi \propto \mathrm{e}^{-\mathcal{F}} \qquad (\mathcal{F}\colon \mathbf{R}^d \to \mathbf{R})$$

CBS interacting particle system

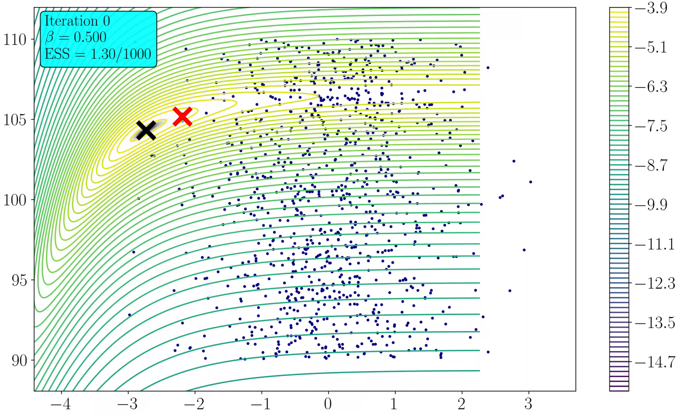
$$\mathrm{d}X_t^j = -\left(X_t^j - \mathcal{M}_\beta\left(\mu_t^J\right)\right)\mathrm{d}t + \sqrt{2(1+\beta)\,\mathcal{C}_\beta(\mu_t^J)}\,\mathrm{d}W_t^j, \qquad j = 1, \dots, J,$$

- β is "inverse temperature" parameter.
- μ_t^J is empirical measure $\mu_t^J = \frac{1}{J}\sum_{j=1}^J \delta_{X_t^j}$.
- $\mathcal{C}_\beta\colon \mathcal{P}(\mathbf{R}^d) \to \mathbf{R}^{d\times d}$ is weighted covariance operator:

$$\mathcal{C}_\beta(\mu) = \frac{\int (x \otimes x)\,\mathrm{e}^{-\beta\mathcal{F}(x)}\,\mu(\mathrm{d}x)}{\int \mathrm{e}^{-\beta\mathcal{F}(x)}\,\mu(\mathrm{d}x)} - \mathcal{M}_\beta(\mu) \otimes \mathcal{M}_\beta(\mu).$$

[1] J. A. Carrillo, F. Hoffmann, A. M. Stuart, and UV. Stud. Appl. Math., 2022.

Illustration



Mean field limits

Taking formally $J \to \infty$ in the interacting particle systems leads to

CBO mean field limit

$$\begin{cases} \mathrm{d}\overline{X}_t = -\left(\overline{X}_t - \mathcal{M}_\beta(\overline{\rho}_t)\right)\mathrm{d}t + \sqrt{2}\sigma\left|\overline{X}_t - \mathcal{M}_\beta(\overline{\rho}_t)\right|\mathrm{d}\overline{W}_t, \\ \overline{\rho}_t = \mathrm{Law}(\overline{X}_t). \end{cases}$$

CBS mean field limit

$$\begin{cases} \mathrm{d}\overline{X}_t = -\left(\overline{X}_t - \mathcal{M}_\beta(\overline{\rho}_t)\right)\mathrm{d}t + \sqrt{2(1+\beta)\mathcal{C}_\beta(\overline{\rho}_t)}\,\mathrm{d}\overline{W}_t, \\ \overline{\rho}_t = \mathrm{Law}(\overline{X}_t). \end{cases}$$

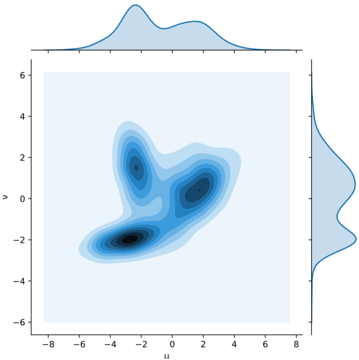
- Nonlinear Markov processes in $\mathbf{R}^d$: future depends on $\overline{X}_t$ and its distribution;
- Associated Fokker–Planck equations are nonlinear and nonlocal.

Notation: Wasserstein distances[1]

Wasserstein distance in $\mathbf{R}^d$ (here $|\cdot|$ is always the Euclidean norm)

For $\mu,\nu \in \mathcal{P}_p(\mathbf{R}^d)$, $\qquad \mathcal{W}_p(\mu,\nu) = \inf_{\gamma\in\Pi(\mu,\nu)} \Big(\mathbf{E}_{(X,Y)\sim\gamma}|X-Y|^p\Big)^{\frac{1}{p}}$

Here $\Pi(\mu,\nu) = \{\gamma \in \mathcal{P}(\mathbf{R}^d\times\mathbf{R}^d) : \mathrm{proj}^x_\sharp\,\gamma = \mu,\ \mathrm{proj}^y_\sharp\,\gamma = \nu\}$.



[1] L.-P. Chaintron and A. Diez. Kinet. Relat. Models, 2022.

Convergence results in mean field law for CBO and CBS

Recall $\mathcal{W}_2 \colon \mathcal{P}_2(\mathbf{R}^d) \times \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ denotes the Wasserstein-2 metric.

Convergence of mean field CBO[1,2]

Under mild conditions including existence of a unique minimizer x_*, there exists $\lambda > 0$ and $x_\beta \in \mathbf{R}^d$ such that

$$\forall t \geqslant 0, \qquad \mathcal{W}_2(\overline{\rho}_t, \delta_{x_\beta}) \lesssim \mathcal{W}_2(\overline{\rho}_0, \delta_{x_\beta})\, \mathrm{e}^{-\lambda t}, \qquad x_* = \arg\min_{\mathbf{R}^d} \mathcal{F}.$$

Furthermore $x_\beta \to x_*$ in the limit $\beta \to \infty$.

Convergence of mean field CBS[3]

If $\pi \propto \mathrm{e}^{-\mathcal{F}}$ is Gaussian and $\overline{\rho}_0$ is Gaussian, then

$$\forall t \geqslant 0, \qquad \mathcal{W}_2(\overline{\rho}_t, \pi) \leqslant C\, \mathrm{e}^{-\left(\frac{\beta}{1+\beta}\right)t}.$$

[1] J. A. Carrillo, Y.-P. Choi, C. Totzeck, and O. Tse. Mathematical Models and Methods in Applied Sciences, 2018.

[2] M. Fornasier, T. Klock, and K. Riedl. SIAM J. Optim., 2024.

[3] J. A. Carrillo, F. Hoffmann, A. M. Stuart, and UV. Stud. Appl. Math., 2022.

Key characteristics of consensus based optimization

- Derivative-free, making them versatile and widely applicable:

```
from cbx.dynamics import CBO
f = lambda x: x[0]**2 + x[1]**2
x = CBO(f, d=2).optimize()
```

- Can be easily implemented in parallel;
- (For the sampling variant) Affine invariant: convergence rate independent of target;
- Theoretical guarantees for the mean field equations.

Question: how to obtain convergence guarantees in the finite-size setting?

Additional notation: Wasserstein distances[1]

Wasserstein distance in $\mathbf{R}^{dJ}$

For $f^J, g^J \in \mathcal{P}(\mathbf{R}^{dJ})$, $\qquad \mathcal{W}_p(f^J, g^J) = \inf_{\gamma \in \Pi(f^J, g^J)} \left(\mathbf{E}_{(\mathbf{X},\mathbf{Y})\sim\gamma} \frac{1}{J} \sum_{j=1}^{J} |X^j - Y^j|^p \right)^{\frac{1}{p}}$

- With this normalization, $\mathcal{W}_p(\mu^{\otimes J}, \nu^{\otimes J}) = \mathcal{W}_p(\mu, \nu)$.
- For associated empirical measures, $\mathbf{E}\left[\mathcal{W}_p(\mu_f^J, \mu_g^J)^p\right] \leqslant \mathcal{W}_p(f^J, g^J)^p$.

In our setting,

- $f^J, \overline{f}^J \in \mathcal{P}(\mathbf{R}^{dJ})$ are joint laws
- $\mu^J, \overline{\mu}^J$ are empirical measures, with laws in $\in \mathcal{P}(\mathcal{P}(\mathbf{R}^d))$

$$\mu^J = \frac{1}{J} \sum_{j=1}^{J} \delta_{X^j}, \qquad \overline{\mu}^J = \frac{1}{J} \sum_{j=1}^{J} \delta_{\overline{X}^j}.$$

[1] L.-P. Chaintron and A. Diez. Kinet. Relat. Models, 2022.

Convergence for the interacting particle systems

Let $f_t^J = \mathrm{Law}\big(X_t^1, \dots, X_t^J\big)$. By the triangle inequality,

$$\mathcal{W}_2\Big(f_t^J, \nu^{\otimes J}\Big) \leqslant \underbrace{\mathcal{W}_2\Big(f_t^J, \overline{\rho}_t^{\otimes J}\Big)}_{\to 0 \text{ as } J \to \infty ???} + \underbrace{\mathcal{W}_2(\overline{\rho}_t, \nu)}_{\leqslant C \, \mathrm{e}^{-\lambda t}}, \qquad \nu = \begin{cases} \delta_{x_\beta} & \text{for CBO,} \\ \mathrm{e}^{-\mathcal{F}} & \text{for CBS.} \end{cases}$$

Pre-existing mean field results for CBO (i.i.d. initial condition and fixed t)

- [1] Based on a compactness argument, it was shown that

$$\mu_t^J \xrightarrow[J \to \infty]{\mathrm{Law}} \overline{\rho}_t, \qquad \text{(no rate)}, \qquad \mu_t^J := \frac{1}{J} \sum_{j=1}^{J} \delta_{X_t^j}.$$

- [2] For all $\varepsilon > 0$ and $t \geqslant 0$, there is $C_\varepsilon > 0$ such that for all J there is $\Omega_\varepsilon \subset \Omega$ satisfying

$$\mathbf{P}[\Omega \setminus \Omega_\varepsilon] \leqslant \varepsilon \quad \text{and} \quad \mathbf{E}\left[\mathcal{W}_2\Big(f_t^J, \overline{\rho}_t^{\otimes J}\Big) \,\Big|\, \Omega_\varepsilon\right] \leqslant C_\varepsilon J^{-\frac{1}{2}}, \qquad C_\varepsilon \xrightarrow[\varepsilon \to 0]{} \infty.$$

Our goal: obtain an estimate of the form $\sup_{t \geqslant 0} \mathcal{W}_p\Big(f_t^J, \overline{\rho}_t^{\otimes J}\Big) \leqslant C J^{-\frac{1}{2}}$.

[1] H. Huang and J. Qiu. Math. Methods Appl. Sci., 2022.
[2] M. Fornasier, T. Klock, and K. Riedl. SIAM J. Optim., 2024.

Outline

Toy example (with $\mathcal{M}(\mu)$ the usual mean under μ)

Interacting particle system:

$$\mathrm{d}X_t^j = -\left(X_t^j - \mathcal{M}\left(\mu_t^J\right)\right)\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}W_t^j, \qquad X_0^j = x_0^j \overset{\text{i.i.d.}}{\sim} \overline{\rho}_0 \qquad j = 1, \dots, J.$$

Mean field limit:

$$\begin{cases} \mathrm{d}\overline{X}_t = -\left(\overline{X}_t - \mathcal{M}(\overline{\rho}_t)\right)\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}\overline{W}_t, \\ \overline{\rho}_t = \mathrm{Law}(\overline{X}_t). \end{cases}$$

Synchronous coupling

We couple to the particle system J copies of the mean field dynamics:

$$\mathrm{d}X_t^j = -\left(X_t^j - \mathcal{M}\left(\mu_t^J\right)\right)\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}W_t^j, \qquad X_0^j = x_0^j, \qquad j = 1, \dots, J,$$
$$\mathrm{d}\overline{X}_t^j = -\left(\overline{X}_t^j - \mathcal{M}(\overline{\rho}_t)\right)\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}W_t^j, \qquad \overline{X}_0^j = x_0^j, \qquad j = 1, \dots, J,$$

with the same initial condition and driving Browian motions.

Using the synchronously coupled to prove propagation of chaos

Synchronous coupling $j \in \{1, \ldots, J\}$

$$\mathrm{d}X_t^j = -\left(X_t^j - \mathcal{M}\left(\mu_t^J\right)\right)\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}W_t^j, \qquad X_0^j = x_0^j,$$
$$\mathrm{d}\overline{X}_t^j = -\left(\overline{X}_t^j - \mathcal{M}(\overline{\rho}_t)\right)\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}W_t^j, \qquad \overline{X}_0^j = x_0^j.$$

Key fact: mean field processes are i.i.d. with law $\overline{X}_t^j \sim \overline{\rho}_t$, so

$$\mathcal{W}_2\left(f_t^J, \overline{\rho}_t^{\otimes J}\right) = \mathcal{W}_2\left(f_t^J, \overline{f}_t^J\right), \qquad \overline{f}_t^J = \mathrm{Law}\left(\overline{X}_t^1, \ldots, \overline{X}_t^J\right).$$

By definition of Wasserstein distance and exchangeability,

$$\mathcal{W}_2\left(f_t^J, \overline{f}_t^J\right)^2 \leqslant \mathbf{E}\left[\frac{1}{J}\sum_{j=1}^{J}\left|X_t^j - \overline{X}_t^j\right|^2\right] = \mathbf{E}\left[\left|X_t^1 - \overline{X}_t^1\right|^2\right].$$

Bounding the remaining term (using Sznitman's approach[1])

Synchronous coupling $j \in \{1, \dots, J\}$

$$
\begin{aligned}
\mathrm{d}X_t^j &= -\left(X_t^j - \mathcal{M}\left(\mu_t^J\right)\right)\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}W_t^j, \qquad & X_0^j &= x_0^j,\\
\mathrm{d}\overline{X}_t^j &= -\left(\overline{X}_t^j - \mathcal{M}(\overline{\rho}_t)\right)\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}W_t^j, \qquad & \overline{X}_0^j &= x_0^j.
\end{aligned}
$$

Key Lemma: Lipschitz continuity of $\mathcal{M}\colon \mathcal{P}_1(\mathbf{R}^d) \to \mathbf{R}^d$

$$
\forall (\mu,\nu) \in \mathcal{P}_1(\mathbf{R}^d) \times \mathcal{P}_1(\mathbf{R}^d), \qquad \left|\mathcal{M}(\mu) - \mathcal{M}(\nu)\right| \leqslant \mathcal{W}_1(\mu,\nu).
$$

$$
\begin{aligned}
\mathbf{E}\left[\left|X_t^1 - \overline{X}_t^1\right|^2\right] &\lesssim \int_0^t \mathbf{E}\left|X_s^1 - \overline{X}_s^1\right|^2 + \mathbf{E}\left|\mathcal{M}\left(\mu_s^J\right) - \mathcal{M}\left(\overline{\rho}_s\right)\right|^2 \mathrm{d}s\\
&\lesssim \int_0^t \mathbf{E}\left|X_s^1 - \overline{X}_s^1\right|^2 + \mathbf{E}\left|\mathcal{M}\left(\mu_s^J\right) - \mathcal{M}\left(\overline{\mu}_s^J\right)\right|^2 + \mathbf{E}\left|\mathcal{M}\left(\overline{\mu}_s^J\right) - \mathcal{M}\left(\overline{\rho}_s\right)\right|^2 \mathrm{d}s\\
&\lesssim \int_0^t \mathbf{E}\left|X_s^1 - \overline{X}_s^1\right|^2 + \mathbf{E}\left[\mathcal{W}_2\left(\mu_s^J, \overline{\mu}_s^J\right)^2\right] \mathrm{d}s + C_{\mathrm{MC}} J^{-1}\\
&\lesssim \int_0^t \mathbf{E}\left|X_s^1 - \overline{X}_s^1\right|^2 \mathrm{d}s + C_{\mathrm{MC}} J^{-1} \quad \overset{\text{Grönwall}}{\rightsquigarrow} \quad \mathbf{E}\left[|X_t^1 - \overline{X}_t^1|^2\right] \leqslant C(t) J^{-1}.
\end{aligned}
$$

[1] A.-S. Sznitman. In École d'Été de Probabilités de Saint-Flour XIX—1989. Springer, Berlin, 1991.

Infinite-dimensional chaos

Sznitman's approach can be generalized to $\mathcal{W}_p$, leading to

$$\mathcal{W}_p\left(f_t^J, \overline{\rho}_t^{\otimes J}\right) = \mathcal{O}\left(\frac{1}{\sqrt{J}}\right) \qquad \text{as } J \to \infty.$$

Question: Can we say anything about the convergence of the empirical measure μ_t^J?

$$\begin{aligned}
\left(\mathbf{E}\mathcal{W}_p\left(\mu_t^J, \overline{\rho}_t\right)^p\right)^{\frac{1}{p}} &\leqslant \left(\mathbf{E}\mathcal{W}_p\left(\mu_t^J, \overline{\mu}_t^J\right)^p\right)^{\frac{1}{p}} + \left(\mathbf{E}\mathcal{W}_p\left(\overline{\mu}_t^J, \overline{\rho}_t\right)^p\right)^{\frac{1}{p}} \\
&\leqslant \left(\mathbf{E}\,\frac{1}{J}\sum_{j=1}^{J}\left|X_t^j - \overline{X}_t^j\right|^p\right)^{\frac{1}{p}} + \left(\mathbf{E}\mathcal{W}_p\left(\overline{\mu}_t^J, \overline{\rho}_t\right)^p\right)^{\frac{1}{p}} \\
&\lesssim J^{-\frac{1}{2}} + J^{-\alpha},
\end{aligned}$$

for $\alpha > 0$ depending on dimension[1].

[1] N. Fournier and A. Guillin. Probab. Theory Related Fields, 2015.

Why the classical Sznitman approach fails for CBO/CBS

Synchronous coupling for CBO, $x_0^j \overset{\text{i.i.d.}}{\sim} \overline{\rho}_0$ for $j \in \{1, \dots, J\}$,

$$\mathrm{d}X_t^j = -\left(X_t^j - \mathcal{M}_\beta\left(\mu_t^J\right)\right)\mathrm{d}t + \sqrt{2}\sigma\left|X_t^j - \mathcal{M}_\beta\left(\mu_t^J\right)\right|\mathrm{d}W_t^j, \qquad X_0^j = x_0^j.$$

$$\mathrm{d}\overline{X}_t^j = -\left(\overline{X}_t^j - \mathcal{M}_\beta(\overline{\rho}_t)\right)\mathrm{d}t + \sqrt{2}\sigma\left|\overline{X}_t^j - \mathcal{M}_\beta(\overline{\rho}_t)\right|\mathrm{d}W_t^j, \qquad \overline{X}_0^j = x_0^j.$$

Technical difficulties:

- $\mathcal{M}_\beta \colon \mathcal{P}_1(\mathbf{R}^d) \to \mathbf{R}^d$ is **not globally Lipschitz** continuous in general:

Example

Take $f \colon \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$, with $\mu_n = \frac{1}{n}\delta_0 + (1 - \frac{1}{n})\delta_n$ and $\nu_n = \delta_n$. Then

$$\mathcal{M}_\beta(\mu_n) \approx 0\,, \qquad \mathcal{M}_\beta(\nu_n) = n\,, \qquad \mathcal{W}_1(\mu_n, \nu_n) = 1\,.$$

- Presence of multiplicative noise that depends on μ_t^J.
- Usual Monte Carlo estimates do not enable to bound

$$\mathbf{E}\left|\mathcal{M}_\beta\left(\overline{\mu}_s^J\right) - \mathcal{M}_\beta(\overline{\rho}_s)\right|^2\,.$$

Outline

Main result: quantitative mean field limits

Assumption (focusing on the unbounded $\mathcal{F}$ setting for simplicity here)

- **Local Lischitz continuity.** $\mathcal{F}$ is bounded from below by $\mathcal{F}_\star = \inf \mathcal{F}$ and satisfies

$$\forall x, y \in \mathbf{R}^d, \qquad |\mathcal{F}(x) - \mathcal{F}(y)| \leq L_f \bigl(1 + |x| + |y|\bigr)^s |x - y|, \qquad s \geqslant 0.$$

- **Growth at infinity**. There are constants $c, u > 0$ and a compact $K \subset \mathbf{R}^d$ such that

$$\forall x \in \mathbf{R}^d \setminus K, \qquad \frac{1}{c}|x|^u \leqslant \mathcal{F}(x) \leqslant c|x|^u.$$

Main theorem[1], holds for both CBO and CBS

If $\mathcal{F}$ satisfies the above assumption and $\overline{\rho}_0$ has infinitely many moments, then

$$\forall J \in \mathbf{N}^+, \qquad \forall j \in \{1, \dots, J\}, \qquad \mathbf{E}\left[\sup_{t \in [0,T]} \left|X_t^j - \overline{X}_t^j\right|^p\right] \leq C J^{-\frac{p}{2}}.$$

[1] N. J. Gerber, F. Hoffmann, and UV. ESAIM Control Optim. Calc. Var., 2025.

Convergence of the weighted mean for i.i.d. samples

Proposition[1,2]

Assume first that $\mathcal{F}$ is bounded. Take $\mu \in \mathcal{P}(\mathbf{R}^d)$ and let

$$\rho = \frac{\left(\int_{\mathbf{R}^d} \mathrm{e}^{-\beta\mathcal{F}} \, \mathrm{d}\mu\right)^2}{\int_{\mathbf{R}^d} \mathrm{e}^{-2\beta\mathcal{F}} \, \mathrm{d}\mu} \in (0,1], \qquad \overline{\mu}^J := \frac{1}{J}\sum_{j=1}^{J} \delta_{\overline{X}^j} \qquad \overline{X}^j \overset{\text{i.i.d.}}{\sim} \mu .$$

which measures the fraction of samples contributing to the weighted mean. Then

$$\sup_{\|\phi\|_{L^\infty} \leqslant 1} \mathbf{E}\left|\frac{\int_{\mathbf{R}^d} \phi \, \mathrm{e}^{-\beta\mathcal{F}} \, \mathrm{d}\overline{\mu}^J}{\int_{\mathbf{R}^d} \mathrm{e}^{-\beta\mathcal{F}} \, \mathrm{d}\overline{\mu}^J} - \frac{\int_{\mathbf{R}^d} \phi \, \mathrm{e}^{-\beta\mathcal{F}} \, \overline{\mu}}{\int_{\mathbf{R}^d} \mathrm{e}^{-\beta\mathcal{F}} \, \mathrm{d}\overline{\mu}}\right|^2 \leqslant \frac{4}{\rho J} .$$

This can be extended to $p \geqslant 2$ and unbounded $\mathcal{F}$ under moment conditions on μ:

$$\rightsquigarrow \quad \mathbf{E}\left|\mathcal{M}_\beta\left(\overline{\mu}^J\right) - \mathcal{M}_\beta(\mu)\right|^p \lesssim \mathbf{E}\left|\overline{X}^1 - \mathbf{E}\overline{X}^1\right|^p J^{-\frac{p}{2}} .$$

[1] P. Doukhan and G. Lang. Bernoulli, 2009.

[2] S. Agapiou, O. Papaspiliopoulos, D. Sanz-Alonso, and A. M. Stuart. Statist. Sci., 2017.

Main ingredients of the proof

Local Lipschitz continuity for $\mathcal{M}_\beta$

For all $p \geqslant 1$, there exists L depending only on $\mathcal{W}_p(\mu, \delta_0)$ and $\mathcal{F}, \beta$ such that

$$\forall (\mu, \nu) \in \mathcal{P}_p(\mathbf{R}^d) \times \mathcal{P}_p(\mathbf{R}^d), \qquad \left|\mathcal{M}_\beta(\mu) - \mathcal{M}_\beta(\nu)\right| \leq L\Big(\mathcal{W}_p(\mu, \delta_0)\Big)\, \mathcal{W}_p(\mu, \nu).$$

Idea: we will use this estimate with $\mu = \overline{\mu}_t^J$ and $\nu = \mu_t^J$.

Moment bounds

Suppose $\overline{\rho}_0 \in \mathcal{P}_q(\mathbf{R}^d)$. Then there is $\kappa > 0$ such that

$$\forall J \in \mathbf{N}^+, \qquad \mathbf{E}\left[\sup_{t \in [0,T]} \left|X_t^j\right|^q\right] \vee \mathbf{E}\left[\sup_{t \in [0,T]} \left|\overline{X}_t^j\right|^q\right] \leqslant \kappa.$$

Sketch of the proof: stopping time approach[1] Details of proof

- Local Lipschitz continuity of $\mathcal{M}_\beta$ motivates stopping time

$$\theta_J = \inf\Big\{t \geqslant 0 : \mathcal{W}_p(\overline{\mu}^J_t, \delta_0) \geqslant R\Big\}, \qquad \overline{\mu}^J_t := \frac{1}{J}\sum_{j=1}^J \delta_{\overline{X}^j_t}.$$

- Then decompose

$$\mathbf{E}\left[\left|X^j_t - \overline{X}^j_t\right|^p\right] = \mathbf{E}\left[\left|X^j_t - \overline{X}^j_t\right|^p \mathbf{1}_{\{\theta_J > T\}}\right] + \mathbf{E}\left[\left|X^j_t - \overline{X}^j_t\right|^p \mathbf{1}_{\{\theta_J \leqslant T\}}\right].$$

- First term can be shown to scale as $CJ^{-\frac{p}{2}}$ using classical approach;
- Second term handled as follows ($q > p$):

$$\mathbf{E}\left[\left|X^j_t - \overline{X}^j_t\right|^p \mathbf{1}_{\{\theta_J \leqslant T\}}\right] \leqslant \mathbf{E}\left[\left|X^j_t - \overline{X}^j_t\right|^q\right]^{\frac{p}{q}} \mathbf{P}[\theta_J \leqslant T]^{\frac{q-p}{q}}.$$

  - First factor bounded using moment bounds.
  - Second factor: for sufficiently large R, by generalized Chebyshev inequality,

$$\forall a > 0, \quad \exists C(a) : \qquad \mathbf{P}\left[\theta_J \leqslant T\right] \leqslant C(a) J^{-a}$$

[1] D. J. Higham, X. Mao, and A. M. Stuart. SIAM J. Numer. Anal., 2002.

Outline

Toy example (with $\mathcal{M}(\mu)$ the usual mean under μ)

Interacting particle system:

$$\mathrm{d}X_t^j = -\Big(X_t^j - \mathcal{M}\big(\mu_t^J\big)\Big)\,\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}W_t^j, \qquad X_0^j = x_0^j \overset{\text{i.i.d.}}{\sim} \overline{\rho}_0 \qquad j = 1, \dots, J.$$

Mean field limit:

$$\begin{cases} \mathrm{d}\overline{X}_t = -\Big(\overline{X}_t - \mathcal{M}(\overline{\rho}_t)\Big)\,\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}\overline{W}_t, \\ \overline{\rho}_t = \mathrm{Law}(\overline{X}_t). \end{cases}$$

Moment decay estimate: by Itô's formula,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\left|X_t^j - \mathcal{M}(\mu_t^J)\right|^2 \leqslant -2\mathbf{E}\left|X_t^j - \mathcal{M}(\mu_t^J)\right|^2 + d\,\mathrm{e}^{-2t}$$

$$\overset{\text{Grönwall}}{\leqslant} \left(\mathbf{E}\left|X_0^j - \mathcal{M}(\mu_0^J)\right|^2 + \frac{d}{2}\right)\mathrm{e}^{-2t}.$$

Similarly

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\left|\overline{X}_t - \mathbf{E}\overline{X}_t\right|^2 \leqslant \left(\mathbf{E}\left|\overline{X}_0 - \mathbf{E}\overline{X}_0\right|^2 + \frac{d}{2}\right)\mathrm{e}^{-2t}.$$

Exploiting convexity with synchronous coupling: Malrieu's approach

Synchronous coupling for toy example

$$
\begin{aligned}
\mathrm{d}X_t^j &= -\Bigl(X_t^j - \mathcal{M}\bigl(\mu_t^J\bigr)\Bigr)\,\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}W_t^j, \qquad X_0^j = x_0^j, \qquad j = 1,\dots,J, \\
\mathrm{d}\overline{X}_t^j &= -\Bigl(\overline{X}_t^j - \mathcal{M}(\overline{\rho}_t)\Bigr)\,\mathrm{d}t + \mathrm{e}^{-t}\,\mathrm{d}W_t^j, \qquad \overline{X}_0^j = x_0^j, \qquad j = 1,\dots,J.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2J}\sum_{j=1}^{J}\Bigl|X_t^j - \overline{X}_t^j\Bigr|^2 = &-\frac{1}{J}\sum_{j=1}^{J}\Bigl\langle X_t^j - \overline{X}_t^j, X_t^j - \mathcal{M}(\mu_t^J) - \overline{X}_t^j + \mathcal{M}(\overline{\mu}_t^J)\Bigr\rangle \\
&+ \frac{1}{J}\sum_{j=1}^{J}\Bigl\langle X_t^j - \overline{X}_t^j, \mathcal{M}(\overline{\mu}_t^J) - \mathcal{M}(\overline{\rho}_t)\Bigr\rangle.
\end{aligned}
$$

The first term is nonpositive. By the Cauchy–Schwarz inequality, we obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\Bigl|X_t^1 - \overline{X}_t^1\Bigr|^2 \leqslant \sqrt{\mathbf{E}\Bigl|X_t^1 - \overline{X}_t^1\Bigr|^2}\sqrt{\mathbf{E}\bigl|\mathcal{M}(\overline{\mu}_t^J) - \mathcal{M}(\overline{\rho}_t)\bigr|^2}.
$$

$$
\rightsquigarrow \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\mathbf{E}\Bigl|X_t^1 - \overline{X}_t^1\Bigr|^2} \leqslant \sqrt{\mathbf{E}\bigl|\mathcal{M}(\overline{\mu}_t^J) - \mathcal{M}(\overline{\rho}_t)\bigr|^2} \lesssim \frac{\mathrm{e}^{-t}}{\sqrt{J}}.
$$

Extending Malrieu's approach to CBO

Synchronous coupling for CBO, $j \in \{1, \dots, J\}$

$$\mathrm{d}X_t^j = -\left(X_t^j - \mathcal{M}_\beta\left(\mu_t^J\right)\right)\mathrm{d}t + \sqrt{2}\sigma\left|X_t^j - \mathcal{M}_\beta\left(\mu_t^J\right)\right|\mathrm{d}W_t^j, \qquad X_0^j = x_0^j,$$
$$\mathrm{d}\overline{X}_t^j = -\left(\overline{X}_t^j - \mathcal{M}_\beta(\overline{\rho}_t)\right)\mathrm{d}t + \sqrt{2}\sigma\left|\overline{X}_t^j - \mathcal{M}_\beta(\overline{\rho}_t)\right|\mathrm{d}W_t^j, \qquad \overline{X}_0^j = x_0^j.$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2J}\sum_{j=1}^J\left|X_t^j - \overline{X}_t^j\right|^2 &= -\frac{1}{J}\sum_{j=1}^J\left\langle X_t^j - \overline{X}_t^j, X_t^j - \mathcal{M}(\mu_t^J) - \overline{X}_t^j + \mathcal{M}(\overline{\mu}_t^J)\right\rangle \\ &\quad - \frac{1}{J}\sum_{j=1}^J\left\langle X_t^j - \overline{X}_t^j, \underbrace{\mathcal{M}(\mu_t^J) - \mathcal{M}_\beta(\mu_t^J) - \mathcal{M}(\overline{\mu}_t^J) + \mathcal{M}_\beta(\overline{\mu}_t^J)}_{\text{Small ???}}\right\rangle \\ &\quad + \frac{1}{J}\sum_{j=1}^J\left\langle X_t^j - \overline{X}_t^j, \underbrace{\mathcal{M}_\beta(\overline{\mu}_t^J) - \mathcal{M}_\beta(\overline{\rho}_t)}_{\text{Small when } J \gg 1}\right\rangle + \sigma \dots\end{aligned}$$

Assumption: for simplicity here, we assume

- no noise ($\sigma = 0$);
- $\mathcal{F}$ bounded and globally Lipschitz;
- initialization in a compact set: $x_0^j \sim \overline{\rho}_0$ with $\overline{\rho}_0$ compactly supported.

Auxiliary results

Notation: For a probability measure $\mu \in \mathcal{P}_1(\mathbf{R}^d)$, let

$$\mathfrak{M}_p(\mu) := \int \left|x - \mathcal{M}(\mu)\right|^p \mu(\mathrm{d}x).$$

Key ingredient: Moment estimate for particle system

For all $p > 0$, there exists $\lambda_p > 0$ such that

$$\mathbf{E}\left[\mathfrak{M}_p(\mu_t^J)\right] \leqslant \mathbf{E}\left[\mathfrak{M}_p(\mu_0^J)\right] \mathrm{e}^{-\lambda_p t}, \qquad \mathbf{E}\left|\overline{X}_t - \mathbf{E}\overline{X}_t\right|^p \leqslant \mathbf{E}\left|\overline{X}_0 - \mathbf{E}\overline{X}_0\right|^p \mathrm{e}^{-\lambda_p t}.$$

Key ingredient: Stability of weighted mean

Then there exists $C_{\mathcal{M}} > 0$ such that for all $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\left|\mathcal{M}_\beta(\mu) - \mathcal{M}(\mu) - \mathcal{M}_\beta(\nu) + \mathcal{M}(\nu)\right| \leqslant C_{\mathcal{M}}\left(\sqrt{\mathfrak{M}_2(\mu)} + \sqrt{\mathfrak{M}_2(\nu)}\right)\mathcal{W}_2(\mu, \nu).$$

Auxiliary results

Notation: For a probability measure $\mu \in \mathcal{P}_1(\mathbf{R}^d)$, let

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Then there exists $C_{\mathcal{M}} > 0$ such that for all $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\left| \mathcal{M}_\beta(\mu) - \mathcal{M}(\mu) - \mathcal{M}_\beta(\nu) + \mathcal{M}(\nu) \right| \leqslant C_{\mathcal{M}} \left(\sqrt{\mathfrak{M}_2(\mu)} + \sqrt{\mathfrak{M}_2(\nu)} \right) \mathcal{W}_2(\mu, \nu).$$

Key ingredient: Moment estimate for particle system

For all $p > 0$ and for σ sufficiently small, there exists $\lambda_p > 0$ such that

$$\mathbf{E}\left[\mathfrak{M}_p(\mu_t^J) \right] \leqslant \mathbf{E}\left[\mathfrak{M}_p(\mu_0^J) \right] \mathrm{e}^{-\lambda_p t}, \qquad \mathbf{E}\left| \overline{X}_t - \mathbf{E}\overline{X}_t \right|^p \leqslant \mathbf{E}\left| \overline{X}_0 - \mathbf{E}\overline{X}_0 \right|^p \mathrm{e}^{-\lambda_p t}.$$

$$\rightsquigarrow \qquad \left| \mathcal{M}(\mu_t^J) - \mathcal{M}_\beta(\mu_t^J) - \mathcal{M}(\overline{\mu}_t^J) + \mathcal{M}_\beta(\overline{\mu}_t^J) \right| \lesssim \mathrm{e}^{-\frac{\lambda_2}{2} t} \mathcal{W}_2\left(\mu_t^J, \overline{\mu}_t^J \right)$$

Extra auxiliary result needed in the presence of noise: $\sigma > 0$

Concentration inequalities for the empirical measures μ_t^J and $\overline{\mu}_t^J$

Suppose that $\overline{\rho}_0$ has finite moments of all orders. Then for all $q > 0$, there is $\kappa_0 > 0$ that for all $\kappa \in (0, \kappa_0)$, there exists $C > 0$ independent of J such that

$$\mathbf{P}[\Omega_\kappa] \leqslant C J^{-\frac{q}{2}} \mathfrak{M}_{2q}(\overline{\rho}_0), \qquad \Omega_\kappa := \left\{ \sup_{t \geqslant 0} \mathrm{e}^{\kappa t} \, \mathfrak{M}_2(\mu_t^J) \geqslant \mathbf{E}\Big[\mathfrak{M}_2(\overline{\rho}_0) \Big] + 1 \right\}.$$

and

$$\mathbf{P}\left[\overline{\Omega}_\kappa\right] \leqslant C J^{-\frac{q}{2}} \mathfrak{M}_{2q}(\overline{\rho}_0), \qquad \overline{\Omega}_\kappa := \left\{ \sup_{t \geqslant 0} \mathrm{e}^{\kappa t} \, \mathfrak{M}_2(\overline{\mu}_t^J) \geqslant \mathbf{E}\Big[\mathfrak{M}_2(\overline{\rho}_0) \Big] + 1 \right\}.$$

Decomposing $\Omega = (\Omega_\kappa \cup \overline{\Omega}_\kappa) \bigcup (\Omega_\kappa \cup \overline{\Omega}_\kappa)^{\complement}$ leads to

$$\mathbf{E}\Big[\Big(\mathfrak{M}_2(\mu_t^J) + \mathfrak{M}_2(\overline{\mu}_t^J) \Big) \mathcal{W}_2^2(\mu_t^J, \overline{\mu}_t^J) \Big] \lesssim J^{-\frac{q}{2}} \mathrm{e}^{-\lambda t} + \mathrm{e}^{-\kappa t} \, \mathbf{E}\Big[\mathcal{W}_2^2(\mu_t^J, \overline{\mu}_t^J) \Big].$$

Uniform-in-time propagation of chaos: main result

Main theorem

Suppose that

- function $\mathcal{F}$ is bounded $\underline{f} \leqslant f \leqslant \overline{f}$ and L_f-globally Lipschitz;
- probability distribution $\overline{\rho}_0$ has finite moments of all orders;
- noise coefficient σ is sufficiently small.

Then there exists $C_{\mathrm{MFL}}\big(\beta, \underline{f}, \overline{f}, L_f, \sigma, d\big)$ such that

$$\sup_{t \geqslant 0} \mathbf{E}\left[\left|X_t^1 - \overline{X}_t^1\right|^2\right] \leqslant \frac{C_{\mathrm{MFL}}}{J}.$$

Corollary: long-time error for the interacting particle system

Theorem

Under the same assumptions as in the previous theorem, it holds that

- *There exists a* $\mathbf{R}^d$*-valued random variable* $\mathcal{M}^X$ *such that*

$$\lim_{t \to +\infty} \mathcal{M}(\mu_t^J) = \mathcal{M}^X$$

- *There exists a* $\gamma > 0$ *such that for all* $t \geqslant 0$*,*

$$|X_t^1 - \mathcal{M}^X| \lesssim \mathrm{e}^{-\gamma t} \quad \textit{almost surely}\,, \qquad \mathbf{E}\Big[|X_t^1 - \mathcal{M}^X|^2\Big] \lesssim \mathrm{e}^{-\gamma t}$$

- *Recall that* $\mathcal{M}(\overline{\rho}_t) \to x_\beta$ *as* $t \to +\infty$*. As a corollary of our result, it holds that*

$$\mathbf{E}\Big[|\mathcal{M}^X - x_\beta|^2\Big] \leqslant \frac{C_{\rm MFL}}{J}\,.$$

Future directions:

- Uniform-in-time estimate for Consensus-Based Sampling;
- Discrete-time estimates;
- Improve the (currently exponential) dependence on β ...

Thank you for your attention!

Future directions:

- Uniform-in-time estimate for Consensus-Based Sampling;
- Discrete-time estimates;
- Improve the (currently exponential) dependence on β ...

Thank you for your attention!

Details of the proof: first term (1/2)

- Starting point: the following is an upper bound for $\left|X_t^j - \overline{X}_t^j\right|^p \mathbf{1}_{\{\theta_J > T\}}$:

$$\left|X_{t\wedge\theta_J}^j - \overline{X}_{t\wedge\theta_J}^j\right|^p \leqslant \left|\int_0^{t\wedge\theta_J} b\Big(X_s^j, \mu_s^J\Big) - b\Big(\overline{X}_s^j, \overline{\rho}_s\Big)\, \mathrm{d}s\right|^p + \left|\int_0^{t\wedge\theta_J} \sigma\Big(X_s^j, \mu_s^J\Big) - \sigma\Big(\overline{X}_s^j, \overline{\rho}_s\Big)\, \mathrm{d}W_s\right|^p .$$

- By Doob's optional stopping and Burkholder–Davis–Gundy,

$$\mathbf{E}\left[\sup_{s\in[0,t]} \left|X_{s\wedge\theta_J}^j - \overline{X}_{s\wedge\theta_J}^j\right|^p\right] \leqslant (2T)^{p-1} \mathbf{E} \int_0^{t\wedge\theta_J} \left|b\Big(X_s^j, \mu_s^J\Big) - b\Big(\overline{X}_s^j, \overline{\rho}_s\Big)\right|^p \mathrm{d}s + C_{\mathrm{BDG}} 2^{p-1} T^{\frac{p}{2}-1} \mathbf{E} \int_0^{t\wedge\theta_J} \left\|\sigma\Big(X_s^j, \mu_s^J\Big) - \sigma\Big(\overline{X}_s^j, \overline{\rho}_s\Big)\right\|_{\mathrm{F}}^p \mathrm{d}s =: A_t + B_t .$$

- Both terms handled similarly. For the drift, by the triangle inequality,

$$A_t \lesssim \int_0^t \mathbf{E}\left|b\Big(X_{s\wedge\theta_J}^j, \mu_{s\wedge\theta_J}^J\Big) - b\Big(\overline{X}_{s\wedge\theta_J}^j, \overline{\mu}_{s\wedge\theta_J}^J\Big)\right|^p \mathrm{d}s + \int_0^t \mathbf{E}\left|b\Big(\overline{X}_s^j, \overline{\mu}_s^J\Big) - b\Big(\overline{X}_s^j, \overline{\rho}_s\Big)\right|^p \mathrm{d}s =: A_t^{(1)} + A_t^{(2)} .$$

Details of the proof: first term (2/2)

- In order to bound $A_t^{(1)}$, recall that $b(x,\mu) = -x + \mathcal{M}_\beta(\mu)$, so

$$\mathbf{E}\left|b\left(X^j_{s\wedge\theta_J}, \mu^J_{s\wedge\theta_J}\right) - b\left(\overline{X}^j_{s\wedge\theta_J}, \overline{\mu}^J_{s\wedge\theta_J}\right)\right|^p \lesssim \mathbf{E}\left|X^j_{s\wedge\theta_J} - \overline{X}^j_{s\wedge\theta_J}\right|^p + \mathbf{E}\left|\mathcal{M}_\beta(\mu^J_{s\wedge\theta_J}) - \mathcal{M}_\beta(\overline{\mu}^J_{s\wedge\theta_J})\right|^p$$

By local $\mathcal{W}_p$ Lipschitz continuity of $\mathcal{M}_\beta$ and definition of θ_J,

$$\mathbf{E}\left|\mathcal{M}_\beta(\mu^J_{s\wedge\theta_J}) - \mathcal{M}_\beta(\overline{\mu}^J_{s\wedge\theta_J})\right|^p \lesssim C(R)\ \mathbf{E}\left|\mathcal{W}_p\left(\mu^J_{s\wedge\theta_J}, \overline{\mu}^J_{s\wedge\theta_J}\right)\right|^p \leqslant C(R)\ \mathbf{E}\left|X^j_{s\wedge\theta_J} - \overline{X}^j_{s\wedge\theta_J}\right|^p.$$

- To bound $A_t^{(2)}$, we use the convergence of the weighted mean for i.i.d. samples[1,2]

$$\mathbf{E}\left|b\left(\overline{X}^j_s, \overline{\mu}^J_s\right) - b\left(\overline{X}^j_s, \overline{\rho}_s\right)\right|^p \propto \mathbf{E}\left|\mathcal{M}_\beta\left(\overline{\mu}^J_s\right) - \mathcal{M}_\beta(\overline{\rho}_s)\right|^p \lesssim J^{-\frac{p}{2}}.$$

Combining the above estimates and using Grönwall's lemma,

$$\mathbf{E}\left[\sup_{t\in[0,T]}\left|X^j_{t\wedge\theta_J} - \overline{X}^j_{t\wedge\theta_J}\right|^p\right] \lesssim J^{-\frac{p}{2}}.$$

[1] P. Doukhan and G. Lang. Bernoulli, 2009.
[2] S. Agapiou, O. Papaspiliopoulos, D. Sanz-Alonso, and A. M. Stuart. Statist. Sci., 2017.

Details of the proof: second term

It remains to bound the probability

$$\mathbf{P}[\theta_J(R) \leqslant T] = \mathbf{P}\left[\sup_{t\in[0,T]} \frac{1}{J}\sum_{j=1}^{J} \left|\overline{X}_t^j\right|^p \geqslant R\right]$$
$$\leqslant \mathbf{P}\left[\frac{1}{J}\sum_{j=1}^{J} Z_j \geqslant R\right], \qquad Z_j := \sup_{t\in[0,T]} \left|\overline{X}_t^j\right|^p.$$

Let $X = \frac{1}{J}\sum_{j=1}^{J} Z_j$. By the Marcinkiewicz–Zygmund inequality, it holds for $r \geqslant 2$ that

$$\mathbf{E}\left|X - \mathbf{E}[X]\right|^r \lesssim J^{-r}\mathbf{E}\left[\left(\sum_{j=1}^{J}\left|Z_j - \mathbf{E}[Z_j]\right|^2\right)^{\frac{r}{2}}\right] \leqslant J^{-\frac{r}{2}}\mathbf{E}\left[\left|Z_1 - \mathbf{E}[Z_1]\right|^r\right],$$

where we used Jensen's inequality and exchangeability. If $R > \mathbf{E}[X]$, then

$$\mathbf{P}\left[X \geq R\right] \leqslant \mathbf{P}\left[\left|X - \mathbf{E}[X]\right|^r \geq \left(R - \mathbf{E}[X]\right)^r\right] \leqslant \mathbf{E}\left[\frac{\left|X - \mathbf{E}[X]\right|^r}{\left(R - \mathbf{E}[X]\right)^r}\right] \leqslant \frac{C J^{-\frac{r}{2}}}{\left(R - \mathbf{E}[X]\right)^r},$$

where we used Markov's inequality.