# Computational Stochastic Processes

## Assessed coursework

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This assignment counts for 25% of the total assessment.

**Handed out** : 28 Feb 2020.

**Deadline for handing in** : 23 Mar 2020.

Please submit a document that starts with a declaration that the content is your own work unless referenced appropriately. The coursework should be submitted via Blackboard. Some important details:

1. You may use a programming language and plotting tools of your choice. If you use software that is not easily available online, document the steps required to obtain the software.
2. You may use any software libraries that you find convenient, such as *NumPy* and *SciPy*.
3. For this assignment, feel free to reuse *without mention* any of the code examples presented in class.
4. Return your solutions in the form of a *Jupyter notebook*, a LaTeX report with all code included, or a `zip` or `tar` archive with a combination of both. Marks are allocated for clarity, including code.
5. Ensure that your results are reproducible, for example by fixing the seed with `np.random.seed(0)`.
6. The three problems will be weighed equally in the calculation of the final mark, and in each problem all the non-bonus subquestions are also weighed equally. The bonus points will enter the calculation of the final mark (/40) as follows:

$$m = \left\lfloor \frac{40}{21} \min(21, m_1 + m_2 + m_3 + b_1 + b_2 + b_3) + \frac{1}{2} \right\rfloor,$$

   where $m_i$ (/7) denote the marks for the problems, not including the bonus questions, and $b_i$ (/1) are the marks for the bonus questions. In words, the bonus points are worth 1/7th, the final mark is out of 40, and this mark is rounded to the nearest integer.
7. If you're unclear on the questions or would like some clarification, do not hesitate to ask or email me.

If you decide to use *Python*, favor

```
import numpy as np
import matplotlib.pyplot as plt
np.sum(...)
plt.plot(...)
```

over

```
from numpy import sum
from matplotlib.pyplot import *
sum(...)
plot(...)
```

Though more verbose, this helps prevent naming conflicts (e.g., `sum` is both a built-in function and a *NumPy* function) and it improves readability – it makes it clear what module a non-builtin function comes from.

**Problem 1** (Mean-square stability of a numerical integrator)**.**

Consider the following scalar stochastic differential equation (SDE):

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad X_0 = 1.$$

The $\theta$ Milstein scheme with time step $\Delta t$ is given by

$$X_{n+1}^{\Delta t} = X_n^{\Delta t} + \big(\theta\, b_{n+1} + (1-\theta)\, b_n\big)\,\Delta t + \sigma_n\,\Delta W_n + \frac{1}{2}\sigma_n\sigma_n'\big((\Delta W_n)^2 - \Delta t\big), \tag{1}$$

where $\theta \in [0,1]$, $b_n = b(X_n^{\Delta t})$, $\sigma_n = \sigma(X_n^{\Delta t})$ and $\sigma_n' = \sigma'(X_n^{\Delta t})$. Consider the SDE

$$dX_t = \mu\, X_t\,\mathrm{d}t + \sigma\, X_t\,\mathrm{d}W_t, \qquad \mu,\, \sigma \in \mathbb{C}. \tag{2}$$

1. Obtain a formula for $\mathbb{E}[|X_t|^2] = \mathbb{E}[X_t\overline{X_t}]$ where $\overline{X_t}$ denotes the complex conjugate of $X_t$. Show that $X_t$ is mean-square stable provided that

$$2\,\mathrm{Re}(\mu) + |\sigma|^2 < 0.$$

2. Show that the $\theta$ Milstein scheme, when applied to the test equation (2), can be written in the form

$$X_{n+1}^{\Delta t} = G(\Delta t, \Delta W_n, \mu, \sigma, \theta) X_n^{\Delta t}$$

and obtain a formula for $G(\Delta t, \mu, \sigma, \theta)$. Let $Z_n^{\Delta t} = \mathbb{E}[|X_n^{\Delta t}|^2]$. Use the previous calculation to obtain an equation of the form

$$Z_{n+1}^{\Delta t} = R(\Delta t, \mu, \sigma, \theta)\, Z_n^{\Delta t}.$$

3. Investigate the region of mean-square stability for the $\theta$ Milstein scheme in the case where $\mu, \sigma \in \mathbb{R}$. Plot the stability region for $\theta = 0, .25, .5, .75, 1$.

4. A numerical scheme is said to be (mean-square) *A-stable* provided that it is mean-square stable for any choice of parameters $\mu$, $\sigma$ such that the test problem (1) is mean-square stable and any $\Delta t > 0$. Are there values of $\theta$ for which the $\theta$ Milstein scheme is A-stable?

5. For $\theta = .25$, $\mu = -1$ and $\sigma = 1$, show that the $\theta$ Milstein scheme is mean-square stable if and only if the time step satisfies $\Delta t < \Delta t^* := 1$. To verify this, implement the method for these parameters and generate $10^5$ replicas of the numerical solution over 100 time steps, with $\Delta t = 2\Delta t^*$ and $\Delta t = \Delta t^*/2$. In both cases, print an estimation of $\mathbb{E}|X_{100}^{\Delta t}|^2$ and comment.

+1 **(Bonus question)**: Consider the weak Euler $\theta$ method:

$$X_{n+1}^{\Delta t} = X_n^{\Delta t} + \big(\theta\, b_{n+1} + (1-\theta)\, b_n\big)\,\Delta t + \sigma_n\,\xi_n\,\sqrt{\Delta t}, \tag{3}$$

with the same notations as in (1) and where $\xi_n$ are i.i.d. random variables coming from a two point point distribution with $\mathbb{P}[\xi_n = 1] = \mathbb{P}[\xi_n = -1] = 1/2$. Calculate the region of asymptotic stability for this scheme in the case of the test equation (2) and comment on your results. (You do not need to examine the limiting case, which requires employing the law of iterated logarithms.)

**Problem 2** (Inference for SDEs). Consider the *Ornstein–Uhlenbeck* equation

$$\mathrm{d}X_t = -\theta(X_t - \mu)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \qquad X_0 = 1 + x, \tag{4}$$

with the parameters $\mu = -1$, $\theta = 1$, $\sigma = \sqrt{2}$ and where $x \sim U(0,1)$.

1. Calculate the solution to (4) analytically and compute $\mathbb{E}[X_T^2]$ for $T = 1$.

2. Show that, for any smooth function $f \in C^\infty([0,T])$ and any $T \geq 0$,

$$I := \int_0^T f(s)\,\mathrm{d}W_s \sim \mathcal{N}\left(0, \int_0^T |f(s)|^2 \mathrm{d}s\right). \tag{5}$$

   You may find it useful to use the notations

$$f_N(s) = f\left(\left\lfloor \frac{s}{\Delta_N} \right\rfloor \Delta_N\right), \qquad \Delta_N = T/N, \qquad I_N = \int_0^T f_N(s)\,\mathrm{d}W_s.$$

   You can take for granted that

   - As $N \to \infty$ it holds that $f_N \to f$ in $L^2([0,T])$, because $f$ is smooth;
   - The sum of normally distributed random variables is normally distributed;
   - Convergence of a sequence of random variables in $L^2(\Omega)$ implies convergence in distribution to the same limit.

3. Using your answers to the previous questions, devise an iterative numerical scheme for (4) of the from

$$X_{n+1}^{\Delta t} = \mu + a(\Delta t)\,(X_n^{\Delta t} - \mu) + b(\Delta t)\,\xi, \qquad \xi \sim \mathcal{N}(0,1), \tag{6}$$

   such that the associated weak error is zero for any observable.

4. Generate $10^5$ replicas of the numerical solution with $\Delta t = .01$ and $T = 1$, and calculate a 99% confidence interval for $E := \mathbb{E}[X_T^2]$ based on these. Comment your results and plot 20 trajectories of the numerical solution.

5. Let $\hat{X} = (X_{t_0}, \dots, X_{t_N})$, with $t_k = k\,\Delta t$ and $\Delta t = .1$, be discrete observations of the solution to (4). Write down the probability distribution function (PDF) of $\hat{X} \in \mathbb{R}^{N+1}$.

6. Use the numerical scheme (6) to generate one long sequence of discrete observations with $\Delta t = .1$ and $N = 10^7$, and calculate numerically the value of the maximum likelihood estimator (MLE) for $\theta$ based on this sequence. To this end, assume that the values of $\mu$ and $\sigma$ are known.

7. Assume now that *prior knowledge* is available for the drift coefficient $\theta$, considered unknown for the sake of this exercise, that was employed to generate the data: $\theta$ was drawn from $\mathcal{N}(2,1)$. Write down the joint PDF $f_{\theta,\hat{X}}(\vartheta, x_0, \dots, x_N)$ of $(\theta, \hat{X}_0, \dots, \hat{X}_N)$, viewed as a random vector in $\mathbb{R}^{N+2}$, as well as the conditional probability distribution $f_{\theta|\hat{X}}(\vartheta|x_0, \dots, x_N)$ of $\theta$ given the discrete observations $(\hat{X}_0, \dots, \hat{X}_N)$. You do not need to calculate the integral

$$I_N(x_0, \dots, x_N) := \int_{\mathbb{R}} f_{\theta,\hat{X}}(\vartheta, x_0, \dots, x_N)\,\mathrm{d}\vartheta.$$

   Plot, up to a constant factor, the conditional distribution of $\theta$ given the observations generated in the previous item. In Bayesian terms, this distribution is known as the *posterior distribution.* The *maximum a posteriori estimator*, usually abbreviated MAP, is defined as the maximizer of the posterior distribution. Calculate the value of this estimator and compare it with that of the MLE.

+1 **(Bonus question)** Based on the posterior distribution found above, calculate a 99% confidence interval for $\theta$ and comment.

The following *Python* functions may be useful for the last two items:

- `scipy.interpolate.interp1d` for interpolation.
- `scipy.integrate.solve_ivp` for solving ordinary differential equations / integrals.
- `scipy.optimize.fminbound` for finding the minimum of a function.
- `scipy.optimize.brentq` for finding the root of a function.

**Problem 3** (Numerical method for a Stratonovich SDE)**.** The Stratonovich integral of a stochastic process $X_t$ with respect to a Brownian motion $W_t$, denoted by

$$\int_0^T X_t \circ \mathrm{d}W_t,$$

is defined as the limit in mean-square as $N \to \infty$ of

$$\sum_{j=0}^{N-1} \frac{X_{t_j^N} + X_{t_{j+1}^N}}{2} \, (W_{t_{j+1}^N} - W_{t_j^N}), \qquad t_j^N = j\, \Delta t^N, \qquad \Delta t^N = \frac{T}{N}.$$

1. Let $X_t$ be the stochastic process given by

$$X_t = \int_0^t b(s,\omega)\, \mathrm{d}s + \int_0^t \sigma(s,\omega) \circ \mathrm{d}W_s,$$

For any twice continuously differentiable function $h : \mathbb{R} \to \mathbb{R}$, it can be shown that $Y_t := h(X_t)$ satisfies

$$Y_t - Y_0 = \int_0^t h'(X_s)\, b(s,\omega)\, \mathrm{d}s + \int_0^t h'(X_s)\, \sigma(s,\omega) \circ \mathrm{d}W_s, \tag{7}$$

like the chain rule of classical calculus – this is the Stratonovich counterpart of Itô's formula. Show this for the function $h(x) = x^2$ in the particular case where $b(t,\omega) = 0$ and $\sigma(t,\omega) = 1$, i.e. show that

$$\int_0^t W_s \circ \mathrm{d}W_s = \frac{W_t^2}{2},$$

based on the definition of the Stratonovich integral. Using (7), calculate

$$\int_0^t W_s^m \circ \mathrm{d}W_s, \qquad m \in \mathbb{N}_{>0}.$$

2. Calculate the solution of the *Stratonovich SDE*

$$\mathrm{d}X_t = \mu\, X_t\, \mathrm{d}t + \sigma\, X_t \circ \mathrm{d}W_t, \qquad X_0 = 1, \tag{8}$$

which is a short-hand way of writing the integral equation

$$X_t = 1 + \int_0^t \mu\, X_t\, \mathrm{d}t + \int_0^t \sigma\, X_t \circ \mathrm{d}W_t. \tag{9}$$

3. In order to define a numerical scheme for Stratonovich SDEs, we introduce the notation, for any $n \in \mathbb{N}_{>0}$, any multi-index $\alpha \in \{0,1\}^n$ and any two times $s \leq t$,

$$J_\alpha^{s,t} = \int_s^t \int_s^{u_1} \cdots \int_s^{u_{n-1}} 1 \circ \mathrm{d}V_{u_n}^{\alpha_1} \ldots \circ \mathrm{d}V_{u_2}^{\alpha_2} \circ \mathrm{d}V_{u_1}^{\alpha_n},$$

with the convention that $\circ\, \mathrm{d}V_t^0 = \mathrm{d}t$ and $\circ\, \mathrm{d}V_t^1 = \circ\, \mathrm{d}W_t$. Using the fact that (7) holds *mutatis mutandis* for vector-valued Stratonovich SDEs, show that the following identities hold almost surely:

$$\begin{aligned} (t-s)\, J_{(1)}^{s,t} &= J_{(0,1)}^{s,t} + J_{(1,0)}^{s,t}, \\ (t-s)\, J_{(1,1)}^{s,t} &= J_{(1,1,0)}^{s,t} + J_{(1,0,1)}^{s,t} + J_{(0,1,1)}^{s,t}. \end{aligned}$$

4. We now consider the Stratonovich SDE

$$\mathrm{d}X_t = b(X_t)\, \mathrm{d}t + \sigma(X_t) \circ \mathrm{d}W_t, \qquad X_0 = x_0. \tag{10}$$

Like Taylor iterative schemes for Itô SDEs, Taylor methods for Stratonovich SDEs are based on truncated stochastic Taylor expansions. Let us use the notations

$$\mathcal{L}_0 f(x) = b(x)\, f'(x), \qquad \mathcal{L}_1 f(x) = \sigma(x)\, f'(x),$$

and let us also introduce, for any $n \in \mathbb{N}_{>0}$ and any multi-index $\alpha \in \{0,1\}^n$, the notation

$$f_\alpha(x) = \mathcal{L}_{\alpha_1} \dots \mathcal{L}_{\alpha_n} \iota(x), \qquad \text{where } \iota : \mathbb{R} \to \mathbb{R}, x \mapsto x.$$

Show that

$$\begin{aligned} X_t - X_s = {} & J^{s,t}_{(0)}\, f_{(0)}(X_s) + J^{s,t}_{(1)}\, f_{(1)}(X_s) \\ & + J^{s,t}_{(0,0)}\, f_{(0,0)}(X_s) + J^{s,t}_{(0,1)}\, f_{(0,1)}(X_s) + J^{s,t}_{(1,0)}\, f_{(1,0)}(X_s) + J^{s,t}_{(1,1)}\, f_{(1,1)}(X_s) + \cdots , \end{aligned}$$

where "$\cdots$" are integrals of higher multiplicity.

5. In general, it can be shown that the iteration

$$X^{\Delta t}_{n+1} = X^{\Delta t}_n + \sum_{\alpha \in \mathcal{A}_\gamma} J^{t_n, t_{n+1}}_\alpha f_\alpha(X^{\Delta t}_n),$$

where the index set $\mathcal{A}_\gamma$ is defined by

$$\mathcal{A}_\gamma = \left\{ \alpha \in \{0,1\}^n : n > 0 \text{ and } \sum_i (2 - \alpha_i) \leq 2\gamma \right\},$$

defines a numerical scheme for (10) of strong order $\gamma$. Write down the scheme of strong order 2 for (9).

6. Implement the scheme for $\mu = -1$ and $\sigma = 1$ and verify the order of strong convergence via numerical experiments.

+1 **(Bonus question)** Based on your knowledge of the exact solution, find an alternative manner of guessing the numerical scheme obtained above, and show rigorously that the scheme has weak order of convergence 2 for the observable $f(x) = x$.