Variance Reduction by Conditioning

Once again, suppose we wish to estimate $\mathbb{E}[Z]$ for some random variable Z. Clearly, if $Z_c = \mathbb{E}[Z \mid W]$ for some random variable W, then

$$\mathbb{E}[Z_c] = \mathbb{E}[Z].$$

To compute the variance of the random variable Z_c we use the law of total variance

Lemma 1. *Let X and Y be random variables such that variance of Y is finite, then*

$$Var[Y] = \mathbb{E}_X\left[Var[Y \mid X]\right] + Var_X\left(\mathbb{E}\left[Y \mid X\right]\right).$$

Proof.

$$\begin{aligned}\mathrm{Var}[Y] &= \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 \\ &= \mathbb{E}\left[\mathbb{E}[Y^2 \mid X]\right] - \left(\mathbb{E}\left[\mathbb{E}[Y \mid X]\right]\right)^2,\end{aligned}$$

using the law of total expectation. Now

$$\mathbb{E}\left[\mathbb{E}[Y^2 \mid X]\right] = \mathbb{E}\left[\mathrm{Var}[Y \mid X] + \left(\mathbb{E}\left[Y \mid X\right]\right)^2\right].$$

and using the fact that

$$\mathrm{Var}\left[\mathbb{E}[Y \mid X]\right] = \mathbb{E}\left[\mathbb{E}[Y \mid X]^2\right] - \left(\mathbb{E}[\mathbb{E}\left[Y \mid X\right]]\right)^2,$$

then the result follows. □

Applying the above lemma with $Y = Z$ and $X = W$ we obtain:

$$\begin{aligned}\mathrm{Var}[Z] &= \mathrm{Var}(\mathbb{E}(Z \mid W)) + \mathbb{E}\left(\mathrm{Var}\left[Z \mid W\right]\right) \\ &= \mathrm{Var}(Z_c) + \mathbb{E}\left(\mathrm{Var}\left[Z \mid W\right]\right) \geq \mathrm{Var}(Z_c),\end{aligned}$$

Thus by conditioning Z with respect to any random variable W we always get a reduction in variance. This motivates the idea of carefully choosing W so that the conditional expectation is computable and gives a signficant variance reduction.

Example 1. Consider the problem of approximating π, via monte carlo integration, using $\mathbb{E}[Z]$, where $Z = f(U_1, U_2)$ with

$$f(u_1, u_2) = 4\mathbf{1}\{(u_1^2 + u_2^2 < 1\}),$$

and where $U_1, U_2 \sim U(0, 1)$. Take

$$Z_c = \mathbb{E}[Z \mid U_1] = 4\mathbb{P}\left(U_2^2 < 1 - U_1^2 \mid U_1\right) = 4\sqrt{1 - U_1^2}.$$

We'll approximate the reduction of variance using this estimator in the worksheets. ⊘

3. Simulation of Stochastic Processes.

We will now focus on the accurate and efficient simulation of continuous stochastic processes.

We recall the definition:

Definition: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{G})$ a measurable space. A continuous time stochastic process is a collection of random variables $X = \{X_t : t \in T\}$ such that for each fixed $t \in T$, X_t is a random variable from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(E, \mathcal{G})$, where $T = [0, \infty)$ or $T = [0, M]$.
The set Ω is known as the sample space and E is the state space of the stochastic process X_t.

For us, the state space E will usually be $\mathbb{R}^d$ equipped with the σ-algebra of Borel sets.

Note that: a stochastic process X can be viewed as a function of t, ω, or both. We write $X(t)$, $X(t, \omega)$, $X_t(\omega)$, etc. We can view a stochastic process in two ways:

- fix $\omega \in \Omega$ and consider the (non-random) map
  $t \longmapsto X(t, \omega) \in E$, for fixed $\omega \in \Omega$
  → we are looking at the path $\omega(t) := X_t(\omega)$, i.e., we identify the sample space Ω with the set of paths from 0 to T (if $T = [0, M]$).
- fix $t \in T$ and consider the map
  $\omega \longmapsto X(t, \omega) \in E$, for fixed $t \in T$.
  → this is a random variable: snapshot of what is happening to all sample points $\omega \in \Omega$ at the time t.

Definition: The finite dimensional distributions (fdd) of a stochastic process X_t are the distributions of the E^K-valued random variables

$$(X(t_1), X(t_2), \dots, X(t_K))$$

for arbitrary $K \in \mathbb{N}$ and arbitrary times $t_i \in T$, $i = 1, \dots, K$.

$$F(\underline{x}) = \mathbb{P}(X(t_i) \leq x_i, \; i = 1, \dots, K), \quad \underline{x} = (x_1, x_2, \dots, x_K).$$

We say that two stochastic processes X_t and Y_t are (stochastically) equivalent if

$$\mathbb{P}(X_t = Y_t) = 1, \text{ for all } t \in T.$$

In this case, X_t is said to be a version of Y_t (and vice versa)

Note that:

- If two processes are equivalent, then they have the same fdd, the converse is not true.
- One can ask if given a set of distributions, these are the fdd of a stochastic process. This is not true in general
- From experiments or numerical simulations, we can only obtain information about the finite dimensional distribution of the process.

We will mostly consider continuous processes, i.e. processes X_t which have continuous paths. A possible alternative would be jump processes.

Definition 2 (Indistinguishable processes)**.** Two stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are indistinguishable if

$$\mathbb{P}(X_t = Y_t \text{ for all } t \in T) = 1.$$

Remark 1. Indistinguishable processes are clearly also stochastically equivalent, but the converse does not hold. Consider for examples the processes $X_t = 0$ and

$$Y_t = \begin{cases} 1 & \text{if } t = U, \\ 0 & \text{otherwise,} \end{cases}$$

where $U \sim U(0,1)$. ⊘

Stationary stochastic processes

Definition 3. A stochastic process is called (strictly) stationary if all FDDs are invariant under time translation: for all $k \in \mathbb{N}$, for all times $t_i \in T$, and $\{\Gamma_i\}_{i=1}^k \subset \mathcal{G}$,

$$\mathbb{P}(X_{t_1} \in \Gamma_1, \ldots, X_{t_k} \in \Gamma_k) = \mathbb{P}(X_{s+t_1} \in \Gamma_1, \ldots, X_{s+t_k} \in \Gamma_k),$$

for $s > 0$ such that $s + t_i \in T$, for every $i = 1, \ldots k$.

In particular, setting $k = 1$, Definition 3 implies that the law of X_t does not depend of t. Stationary processes therefore describe phenomena which do not change in time. Let X_t be a real-valued random process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite second moment (i.e. $X_t \in L^2(\Omega, \mathbb{P})$ for all $t \in T$). Assume that X_t is strictly stationary. Then

$$\mathbb{E}[X_{t+s}] = \mathbb{E}[X_t], \quad \forall s \in T,$$

from which we conclude that $\mathbb{E}X_t = \mathbb{E}X_0$ is constant, and moreover we have that

$$\mathbb{E}\left[(X_{t_1+s} - \mu)(X_{t_2+s} - \mu)\right] = \mathbb{E}\left[(X_{t_1} - \mu)(X_{t_2} - \mu)\right], \quad \forall s \in T.$$

This implies that the covariance function $C(t,s)$ only depends on the difference $t - s$, that is to say $C(t,s) = C(t-s)$, which motivates the following definition.

Definition 4. A continuous time stochastic process $\{X_t\}_{t \in T}$ is *wide sense stationary* (WSS) or *second-order stationary* or *weakly stationary* if it has finite first and second moments and

1. $\mathbb{E}(X_t)$ is constant, i.e. it does not depend on t;
2. $\operatorname{cov}(X_t, X_s)$ is a function of the difference $t - s$;

The function $C(t-s) = \operatorname{cov}(X_t, X_s)$ is the *autocovariance function* of the process X. Notice that for mean-zero processes, $C(t) = \mathbb{E}(X_t X_0)$, whereas $C(0) = \mathbb{E}X_t^2$, which is finite, by assumption. Since we have assumed that X_t is a real valued process, we have that $C(t) = C(-t)$, $\forall t \in \mathbb{R}$. From the discussion above, it is clear that a strictly stationary $L^2(\Omega)$ random variable is also wide-sense stationary. The converse is not true in general. An exception to this is the case of Gaussian processes:

Lemma 5. *A Gaussian process (see definition below) is strictly stationary if and only if it is weakly stationary.*

Proof. Gaussian distributions are determined by their mean vector and covariance matrix. Since the mean and covariance of a weakly stationary process do not change when the times are shifted, this implies that the finite dimensional distributions are invariant under time shift. □

If its autocovariance function $C(\cdot)$ is absolutely integrable, then a weakly stationary process is ergodic in the following sense:

Theorem 6 ($L^2(\Omega, \mathbb{P})$ (mean-square) ergodic theorem)**.** *Let $\{X_t\}_{t\geq 0}$ be a weakly stationary process with mean μ and autocovariance function $C \in L^1(0,\infty)$, i.e.*

$$\int_0^\infty |C(t)| \, \mathrm{d}t < \infty.$$

Then

$$\lim_{T\to\infty} E_T := \lim_{T\to\infty} \mathbb{E}\left(\left|\frac{1}{T}\int_0^T X_s \, \mathrm{d}s - \mu\right|^2\right) = 0.$$

Proof. Let $I^2 = [0,T]\times[0,T]$. We notice that, by applying Young's inequality and using the fact that X_t is mean-square integrable:

$$\begin{aligned}\int_{I^2}\int_\Omega |X_s(\omega)-\mu|\,|X_t(\omega)-\mu|\,\mathbb{P}(\mathrm{d}\omega)\,\mathrm{d}(s,t) &\leq \frac{1}{2}\int_{I^2}\int_\Omega |X_s(\omega)-\mu|^2 + |X_t(\omega)-\mu|^2\,\mathbb{P}(\mathrm{d}\omega)\,\mathrm{d}(s,t)\\ &= \int_{I^2} C(0)\,\mathrm{d}(s,t) < \infty.\end{aligned}$$

We can therefore apply Fubini–Tonelli's theorem, which implies that

$$\int_{I^2} |X_s(\omega)-\mu|\,|X_t(\omega)-\mu|\,\mathrm{d}(s,t) < \infty \qquad \text{almost surely,} \tag{1}$$

and that the two integrals can be swapped:

$$\int_{I^2}\int_\Omega (X_s(\omega)-\mu)\,(X_t(\omega)-\mu)\,\mathbb{P}(\mathrm{d}\omega)\,\mathrm{d}(s,t) = \int_\Omega\int_{I^2} (X_s(\omega)-\mu)\,(X_t(\omega)-\mu)\,\mathrm{d}(s,t)\,\mathbb{P}(\mathrm{d}\omega).$$

By (1), we can apply Fubini–Tonelli's theorem again to rewrite the integral on the right-hand side as the triple integral

$$\int_\Omega\int_0^T\int_0^T (X_s(\omega)-\mu)\,(X_t(\omega)-\mu)\,\mathrm{d}s\,\mathrm{d}t\,\mathbb{P}(\mathrm{d}\omega) = \int_\Omega \left|\int_0^T (X_s(\omega)-\mu)\,\mathrm{d}s\right|^2 \mathbb{P}(\mathrm{d}\omega) = T^2\,E_T,$$

by definition of E_T. (This level of detail for the application of Fubini–Tonelli's theorem is not necessary, but it was included here for completeness.)

We therefore obtain

$$E_T = \int_\Omega \left|\int_0^T (X_s(\omega)-\mu)\,\mathrm{d}s\right|^2 \mathbb{P}(\mathrm{d}\omega) = \frac{1}{T^2}\int_{I^2} \mathbb{E}\left((X_s-\mu)(X_t-\mu)\right)\,\mathrm{d}(s,t) = \frac{1}{T^2}\int_{I^2} C(t-s)\,\mathrm{d}(s,t).$$

By employing the fact that $C(\cdot)$ is an even function and by using the change of variables $u = t-s$,

$\nu = t$, we obtain

$$E_T = \frac{2}{T^2} \int_0^T \int_0^t C(t-s)\,\mathrm{d}s\,\mathrm{d}t = \frac{2}{T^2} \int_0^T (T-u)\,C(u)\,\mathrm{d}u \leq \frac{2}{T} \int_0^\infty |C(u)|\,\mathrm{d}u,$$

which, by letting $T \to \infty$, allows us to conclude the proof. □

3.1. Gaussian Stochastic Processes

Gaussian stochastic processes are a very important class of continuous-time processes which arise in many applications.

Definition: A one-dimensional Gaussian process is a stochastic process for which $E = \mathbb{R}$ and all its fdd's are Gaussian, i.e., every finite dimensional vector $(X_{t_1}, X_{t_2}, \dots, X_{t_k})$ is a $N(\mu_{t_1 \dots t_k}, C_{t_1, \dots, t_k})$ random variable for some vector $\mu_{t_1, \dots, t_k}$ and a symmetric non-negative definite matrix $C_{t_1, \dots, t_k}$, for all $k \in \mathbb{N}$, $t_1, t_2, \dots, t_k \in \mathbb{R}$.
It is straightforward to extend this definition to arbitrary dimensions.

Note that: a key feature of Gaussian processes is that they are completely characterised by their mean $\boxed{\mu(t) := \mathbb{E} X_t}$ and covariance function

$$\boxed{C(t,s) = \mathbb{E}\left((X_t - \mu(t))(X_s - \mu(s))\right)}.$$

Proposition: For any function $\mu : T \to \mathbb{R}$ and any non-negative definite function $C : T \times T \to \mathbb{R}$, there exists a Gaussian process X_t on T such that

$$\mathbb{E}(X_t) = \mu(t), \quad \mathrm{Cov}(X_t, X_s) = C(t,s).$$

This proposition states that there exists X_t with a given mean and covariance as long as

$$\sum_{i=1}^{k} \sum_{j=1}^{k} C(t_i, t_j)\, c_i \bar{c}_j \geq 0$$

$\forall\, k \in \mathbb{N}$, $t_1, \dots, t_k \in \mathbb{R}$, $c_1, \dots, c_k \in \mathbb{R}$.

examples of Gaussian processes

① Brownian motion:
$\mu(t) = 0$, $C(t,s) = \min(t,s)$

② Brownian bridge $0 \le s, t \le 1$.
$\mu(t) = 0$, $C(t,s) = \min(t,s) - ts$

③ Ornstein-Uhlenbeck Process
$\mu(t) = 0$, $C(t,s) = \lambda \exp(-\alpha|t-s|)$, $\alpha, \lambda \ge 0$

④ Fractional Brownian motion
$\mu(t) = 0$, $C(t,s) = (t^{2H} + s^{2H} - |t-s|^{2H})/2$
where $H \in (0,1)$ is the Hurst parameter.

3.2. Brownian motion, related stochastic processes, and their simulation.

The most well-known Gaussian process is the Brownian motion.

Definition: We define a Wiener Process or Brownian Motion (BM) to be a real-valued stochastic process $(W_t)_{t \ge 0}$ such that

(i) $W_0 = 0$
(ii) W_t is continuous a.s.
(iii) $W_t - W_s \sim N(0, t-s)$ for all $0 \le s \le t$
(iv) W_t has independent increments, i.e., $\forall n \in \mathbb{N}$, $\forall\, 0 \le t_1 < t_2 < \dots < t_n$, $W_{t_1}, W_{t_2 - t_1}, \dots, W_{t_n} - W_{t_{n-1}}$ are independent r.v.

This generalises easily to higher dimensions: a d-dimensional Brownian motion is a d-vector $(W_1(t), \dots, W_d(t))$ of independent, one dimensional BM.

From the definition of BM, it follows that:

(1) W_t is a Gaussian process

(2) $\mu(t) = \mathbb{E}(W_t) = 0, \quad \forall t \geq 0$

(3) $\mathrm{Cov}(W_t, W_s) = \mathbb{E}(W_t W_s) = \min(t, s)$:
suppose $s \leq t$. then
$\mathbb{E}(W_t W_s) = \mathbb{E}((W_t - W_s + W_s) W_s) = \mathbb{E}(W_s^2) = s$.
since $\mathbb{E}((W_t - W_s) W_s) = 0$ (indep. increments)

(4) For all $a \leq b$
$$\mathbb{P}(W_t \in (a,b)) = \frac{1}{2\pi t} \int_a^b \exp(-x^2/2t)\, dx.$$

Note that: the Brownian motion can be seen as the limit of a random walk. A more powerful result is known as "functional central limit theorem" and is as follows:

Theorem (Donsker's theorem). Let $\{X_i\}_{i \geq 0}$ be i.i.d. r.v. with $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_i^2) = 1$. Define
$$S_n = X_1 + X_2 + \ldots + X_n$$
and let $$Z_n(t) = \frac{S_{[nt]}}{\sqrt{n}} \qquad 0 \leq t \leq 1.$$
then $Z_n \xrightarrow{d} W$ where W is a BM in $[0,1]$.

Suppose now that we wish to simulate a Gaussian process X_t with given mean $\mu(t)$ and covariance function $C(t,s)$ at a finite number of time steps $t_0, \ldots, t_N$. By the definition of a Gaussian process, the vector
$$\underline{X} = (X(t_0), X(t_1), \ldots, X(t_N))$$
is a multivariate Gaussian r.v. with
$$\underline{m} = (\mu(t_0), \mu(t_1), \ldots, \mu(t_N))$$
and covariance matrix
$$\Sigma_{ij} = \mathrm{Cov}(X(t_i), X(t_j))$$

Algorithm: Generate a Gaussian Process

1. Generate the mean vector $\underline{m} = (\mu(t_0), \dots, \mu(t_N))$
2. Generate the covariance matrix $\Sigma_{ij} = C(t_i, t_j)$
4. Generate a sample from the distribution
$$\mathcal{N}(\underline{m}, \Sigma), \quad \text{(multivariate Gaussian)}$$

Note that we will need to use the Cholesky decomposition of Σ as mentioned before.

→ Some examples of Gaussian processes ⇒ matlab examples.

Note that:

- If X_t is a Brownian motion, we can take advantage of the fact that it is a Markov process, i.e., you only need to know X_{t_n} to compute $X_{t_{n+1}}$. We can write (exercise)
$$X(t_{n+1}) = X(t_n) + \sqrt{t_{n+1} - t_n}\, Z, \quad Z \sim \mathcal{N}(0,1).$$
- We can also generate samples of other related processes such as the geometric Brownian motion (gBm)
$$S_t = S_0 \exp(at + \sigma W_t)$$
which is important in financial mathematics
- We can derive an update formula for the Ornstein-Uhlenbeck process as well, using a time change:

  Lemma: Let $W(t)$ be a BM. Then the process $V(t) = e^{-t} W(e^{2t})$ is a Gaussian process with mean zero and covariance $C(s,t) = e^{-|t-s|}$.

- Some interesting properties of the fractional BM:
  - If $H = 1/2 \Rightarrow$ BM
  - $\mathbb{E}(W_0^H) = \mathbb{E}(W_t^H) = 0$, $\mathbb{E}(W_t^H)^2 = |t|^{2H}$
  - $\mathbb{E}(W_t^H - W_s^H)^2 = |t-s|^{2H}$
  - $(W_{\alpha t}^H, t \geq 0) = (\alpha^H W_t^H, t \geq 0)$, $\alpha > 0$

Karhunen–Loève expansion

Theorem 7 (Mercer)**.** *Suppose that $C(\cdot,\cdot)$ is a continuous symmetric non-negative definite function on $[a,b]\times[a,b]$, i.e.*

$$\sum_{i=1}^{n}\sum_{i=1}^{n} C(t_i,t_j)\,c_i\,c_j \geq 0 \qquad \text{for all } n\in\mathbf{N} \text{ and } t_i,\ldots,t_n \text{ in } [a,b],$$

and let $K_C : L^2(a,b)\to L^2(a,b)$ be the operator defined by

$$K_C f(t) = \int_a^b C(s,t)\,f(s)\,\mathrm{d}s.$$

Then

- *There is an orthonormal basis $\{\phi_n\}_{n\in\mathbf{N}}$ of $L^2(a,b)$ consisting of eigenfunctions of the operator K_C, the corresponding eigenvalues $\{\lambda_n\}_{n\in\mathbf{N}}$ are nonnegative;*
- *The eigenfunctions corresponding to non-zero eigenvalues are continuous;*
- *$C(\cdot,\cdot)$ admits the representation*

$$C(s,t) = \sum_{n=1}^{\infty} \lambda_n\,\phi_n(s)\,\phi_n(t), \tag{2}$$

where the convergence is absolute and uniform, i.e.

$$\sum_{n=1}^{\infty} |\lambda_n\,\phi_n(s)\,\phi_n(t)|,$$

seen as a sequence of functions on $[a,b]\times[a,b]$, converges uniformly.

The full proof of Mercer's theorem is beyond the scope of this course, so in the partial proof below (included for information purposes) we will restrict our attention to the first two claims. To this end we will admit without proof two auxiliary results:

Theorem 8 (Arzelà–Ascoli)**.** *Consider a sequence of real-valued continuous functions $\{f_n\}_{n\in\mathbf{N}}$, defined on a closed and bounded interval $[a,b]$. If this sequence is* uniformly bounded *and* uniformly equicontinuous*, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbf{N}}$ that converges uniformly.*

Theorem 9 (Spectral theorem for compact self-adjoint operators)**.** *Suppose that $K : H\to H$ is a compact self-adjoint operator on a separable infinite-dimensional Hilbert space. Then there exists a basis $\{\phi_n\}_{n\in\mathbf{N}}$ of consisting of eigenfunctions of K. In addition, the corresponding eigenvalues are real $\lambda_n\to 0$ as $n\to 0$.*

Partial proof of Mercer's theorem. The operator K_C is clearly self-adjoint in $L^2(a,b)$ because, denoting by $\langle\cdot,\cdot\rangle$ the inner product on $L^2(a,b)$ and employing Fubini's theorem, we have

$$\forall f,g\in L^2(a,b), \qquad \langle K_C f, g\rangle = \int_a^b\int_a^b C(s,t)\,f(s)\,g(t)\,\mathrm{d}s\,\mathrm{d}t = \langle f, K_C g\rangle .$$

We now show that K_C is compact, which will enable us to use the spectral theorem for compact self-adjoint operators. To this end, let $\{f_n\}_{n\in\mathbf{N}}$ be sequence of functions bounded in $L^2(a,b)$, i.e. $\|f_n\| \leq A < \infty$ for all n, where $\|\cdot\|$ denotes the norm of $L^2(a,b)$. We will show that the mapped sequence $\{K_C f_n\}_{n\in\mathbf{N}}$ is uniformly bounded and uniformly equicontinuous. Since $C(\cdot,\cdot)$ is continuous on the compact set $[a,b]\times[a,b]$, it admits a maximum $M<\infty$ on that set. Therefore, for all n and all $t\in[a,b]$ it holds that

$$|K_C f_n(t)| \leq \int_a^b |C(s,t)\, f_n(s)|\,\mathrm{d}s \leq M\int_a^b |f_n(s)|\times 1\,\mathrm{d}s \leq M\,\|f_n\|\,\|1\| < M\,A\,\|1\|,$$

which shows the uniform boundedness. (Here we employed the Cauchy–Schwarz inequality.) Let us now denote by $\omega(\cdot)$ the modulus of continuity of $C(\cdot,\cdot)$, i.e. $\omega(\cdot)$ is such that

$$|C(s_1,t_1)-C(s_2,t_2)| \leq \omega(\sqrt{(s_2-s_1)^2+(t_2-t_1)^2}) \qquad \forall (s_1,t_1),(s_2,t_2)\in[a,b]\times[a,b].$$

With this notation, for any $s,t\in[a,b]$ and any n it holds that

$$\begin{aligned} |K_C f_n(t) - K_C f_n(s)| &\leq \int_a^b |C(u,t)-C(u,s)|\,|f_n(u)|\,\mathrm{d}u \\ &\leq \omega(|t-s|)\,\|f_n\|\,\|1\| < \omega(|t-s|)\,A\,\|1\|, \end{aligned} \tag{3}$$

which shows the uniform equicontinuity. By Arzelà–Ascoli, there is therefore a subsequence $\{K_C f_{n_k}\}_{k\in\mathbf{N}}$ that converges uniformly, and thus also in $L^2(a,b)$. We have thus shown that for any bounded sequence in $L^2(a,b)$, the mapped sequence contains a converging (in $L^2(a,b)$) subsequence, which implies that K_C is compact, by definition.

The spectral theorem therefore implies the existence of an orthonormal basis $\{\phi_n\}_{n\in\mathbf{N}}$ of $L^2(a,b)$ consisting of eigenfunctions of K_C. The eigenfunctions corresponding to non-zero eigenvalues are uniformly continuous because, by (3),

$$|\phi_n(t)-\phi_n(s)| = \frac{1}{\lambda_n}|K_C\phi_n(t)-K_C\phi_n(s)| \leq \frac{1}{\lambda_n}\omega(|t-s|)\,A\,\|1\| \qquad \forall s,t\in[a,b].$$

Now notice that

$$\lambda_n = \langle K_C\phi_n,\phi_n\rangle = \int_a^b\int_a^b C(s,t)\,\phi_n(s)\,\phi_n(t)\,\mathrm{d}s\,\mathrm{d}t.$$

Since the integrand is uniformly continuous, the integral can be approximated arbitrary well by a (double) Riemann sum, and this sum is nonnegative by the assumption on C. Therefore, the eigenvalues are nonnegative. □

Now assume that X_t is a centered process ($\mathbb{E}[X_t]=0$) with continuous autocovariance function $C(\cdot,\cdot)$. This latter assumption implies in particular that the paths of X_t are almost surely in $L^2(a,b)$. Indeed

$$\mathbb{E}\left[\int_a^b X_t^2\,\mathrm{d}t\right] = \int_a^b \mathbb{E}[X_t^2]\,\mathrm{d}t = \int_a^b C(t,t)\,\mathrm{d}t < \infty.$$

If follows from this that, given a complete orthonormal basis $\{\phi_n\}_{n\in\mathbf{N}}$ of $L^2(a,b)$, the following

two statements hold almost surely:

$$\textbf{(i)} \qquad Z_i := \int_a^b X_t\,\phi_i(t)\,\mathrm{d}t \le \sqrt{\int_a^b X_t^2\,\mathrm{d}t}\sqrt{\int_a^b \phi_i(t)^2\,\mathrm{d}t} = \sqrt{\int_a^b X_t^2\,\mathrm{d}t} < \infty \qquad \forall i \in \mathbf{N}.$$

$$\textbf{(ii)} \qquad S_n := \sum_{i=1}^n Z_i\,\phi_i(t)\,\mathrm{d}t \to X_t \qquad \text{in } L^2(a,b) \text{ as } n \to \infty.$$

Remark 2 (not examinable). The foregoing discussion shows that $S_n := \sum_{i=1}^n Z_i\phi_i$, viewed as a random variable with values in $L^2(a,b)$, converges to X_t almost surely. On the other hand, notice that

$$\sum_{i=1}^n \|Z_i\phi_i\|^2_{L^2(\Omega\times(a,b))} = \sum_{i=1}^n \mathbb{E}\left[\int_a^b Z_i^2\,\phi_i(t)^2\,\mathrm{d}t\right] = \sum_{i=1}^n \mathbb{E}\left[Z_i^2\right] = \mathbb{E}\left[\sum_{i=1}^n Z_i^2\right].$$

Here $L^2(\Omega\times(a,b))$ denotes as the L^2 space on $\Omega\times(a,b)$ equipped with the product measure $\mathbb{P}\times\lambda$, with λ the Lebesgue measure on (a,b). By Bessel's inequality, and since X_t is in $L^2(a,b)$ almost surely, the sum in the expectation is bounded from above by $\int_a^b X_t^2\,\mathrm{d}t$ almost surely. We deduce

$$\sum_{i=1}^n \|Z_i\phi_i\|^2_{L^2(\Omega\times(a,b))} \le \mathbb{E}\left[\int_a^b X_t^2\,\mathrm{d}t\right] < \infty,$$

and so $S_n = \sum_{i=1}^n Z_i\phi_i$ converges in $L^2(\Omega\times(a,b))$ to a limit Y_t. To show that $Y_t = X_t$, notice that the convergence of S_N to Y_t in $L^2(\Omega\times(a,b))$ implies that $\Delta_n := \|Y_t - S_n\|_{L^2(a,b)}$ (a real-valued random variable) converges in $L^2(\Omega)$ to 0. Therefore (this is a fundamental result in functional analysis) there exists a subsequence Δ_{n_k} that converges to 0 almost surely, i.e. almost surely it holds that $\|Y_t - S_{n_k}\|_{L^2(a,b)} \to 0$ as $k\to\infty$, which shows that $Y_t = X_t$ by uniqueness of the almost sure limit. We conclude that $S_n(t)$ converges to X_t also in $L^2(\Omega\times(a,b))$. The difference

$$\|S_n(t) - X_t\|_{L^2(\Omega\times(a,b))} = \left(\int_a^b \mathbb{E}|S_n(t) - X_t|^2\,\mathrm{d}x\right)^{1/2} = \left(\int_a^b \mathbb{E}\left|\sum_{i=n+1}^\infty Z_i\,\phi_i(t)\right|^2\,\mathrm{d}x\right)^{1/2}.$$

is called the *total mean square error* between S_n and X_t. (Note that the square is inside the expectation, i.e. when we write $\mathbb{E}|\cdot|^2$ we mean $\mathbb{E}[|\cdot|^2]$). ⊘

The Karhunen–Loève theorem shows the convergence of S_n to X_t in a stronger sense.

Theorem 10 (Karhunen–Loève)**.** *Let X_t be a zero-mean square-integrable stochastic process defined over a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and indexed over a closed and bounded interval $[a,b]$. Assume additionally that the autocovariance function of the process, denoted by $C(\cdot,\cdot)$, is continuous on $[a,b]\times[a,b]$. Then $C(\cdot,\cdot)$ satisfies the conditions of Mercer's theorem and, denoting by $\{\phi_n\}_{n\in\mathbf{N}}$ and λ_n the eigenfunctions and corresponding eigenvalues of the operator*

$$K_C : f \to \int_a^b C(s,\cdot)\,f(s)\,\mathrm{d}s,$$

X_t admits the following representation

$$X_t = \sum_{n=1}^{\infty} Z_n \phi_n(t), \qquad Z_n = \int_a^b X_t \, \phi_n(t) \, \mathrm{d}t,$$

where the convergence is in $L^2(\Omega, \mathbb{P})$ uniformly in t. Additionally, the random variables Z_n have zero mean, are uncorrelated and have variance λ_n.

Proof. Below we will make use of Fubini–Tonelli's theorem without explicitly mentioning that we do so. It follows from the definitions of Z_n that

$$\mathbb{E}[Z_i] = \mathbb{E}\left[\int_a^b X_t \, \phi_i(t) \, \mathrm{d}t\right] = \int_a^b \mathbb{E}[X_t] \, \phi_i(t) \, \mathrm{d}t = 0, \qquad \forall i \in \mathbf{N}.$$

Similarly, for all $i, j \in \mathbf{N}$ it holds that

$$\begin{aligned}
\mathbb{E}[Z_i \, Z_j] &= \mathbb{E}\left[\int_a^b X_s \, \phi_i(s) \, \mathrm{d}s \int_a^b X_t \, \phi_j(t) \, \mathrm{d}t\right] = \mathbb{E}\left[\int_a^b \int_a^b X_s \, X_t \, \phi_i(s) \, \phi_j(t) \, \mathrm{d}s \, \mathrm{d}t\right] \\
&= \int_a^b \int_a^b \mathbb{E}[X_s \, X_t] \, \phi_i(s) \, \phi_j(t) \, \mathrm{d}s \, \mathrm{d}t = \int_a^b \int_a^b C(s,t) \, \phi_i(s) \, \phi_j(t) \, \mathrm{d}s \, \mathrm{d}t \\
&= \int_a^b \int_a^b C(s,t) \, \phi_i(s) \, \mathrm{d}s \, \phi_j(t) \, \mathrm{d}t = \int_a^b \lambda_i \, \phi_i(t) \, \phi_j(t) \, \mathrm{d}t = \lambda_i \, \delta_{ij}.
\end{aligned}$$

It remains to show the convergence, for which we will use the third statement in Mercer's theorem, which follows from $\mathbb{E}[Z_i \, Z_j] = \delta_{ij} \, \lambda_i$ and the fact that $\{\phi_n\}_{n \in \mathbb{N}}$ are the eigenfunctions of K_C:

$$\begin{aligned}
\mathbb{E}\left[|X_t - S_n(t)|^2\right] &= \mathbb{E}\left[|X_t|^2 + |S_n(t)|^2 - 2 \, X_t \, S_n(t)\right] \\
&= C(t,t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}[Z_i \, Z_j] \, \phi_i(t) \, \phi_j(t) - 2 \sum_{i=1}^{n} \mathbb{E}[X_t \, Z_i] \, \phi_i(t) \\
&= C(t,t) + \sum_{i=1}^{n} \lambda_i \, \phi_i(t) \, \phi_i(t) - 2 \sum_{i=1}^{n} \mathbb{E}\left[\int_a^b X_t \, X_s \, \phi_i(s) \, \mathrm{d}s\right] \phi_i(t) \\
&= C(t,t) + \sum_{i=1}^{n} \lambda_i \, \phi_i(t) \, \phi_i(t) - 2 \sum_{i=1}^{n} \int_a^b C(s,t) \, \phi_i(s) \, \mathrm{d}s \, \phi_i(t) \\
&= C(t,t) - \sum_{i=1}^{n} \lambda_i \, \phi_i(t) \, \phi_i(t).
\end{aligned}$$

By Mercer's theorem, $C(t,t) - \sum_{i=1}^{n} \lambda_i \, \phi_i(t) \, \phi_i(t)$ converges to 0 uniformly in time as $n \to \infty$, which concludes the proof. □

For Gaussian processes, the coefficients $\{Z_n\}_{n \in \mathbf{N}}$ in the Karhunen–Loève expansion are all normally distributed. This is because the integral of a Gaussian process is a Gaussian random variable (not examinable).

Exercise 1. Let X_t be a Gaussian process on $[a, b]$ with continuous mean $\mu(t)$ and autocovariance $C(s,t)$, and with almost surely Riemann-integrable paths. Show that

$$\int_a^b X_t \, \mathrm{d}t \sim \mathcal{N}\left(\int_a^b \mu(t) \, \mathrm{d}t, \int_a^b \int_a^b C(s,t) \, \mathrm{d}s \, \mathrm{d}t\right).$$

Proof. By assumption, it holds almost surely that

$$I := \int_a^b X_t \, \mathrm{d}t = \lim_{n\to\infty} I_n =: \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{i=0}^{n-1} X_{t_i^n}, \qquad t_i^n := a + \frac{i}{n}(b-a). \tag{4}$$

The argument of the limit on the right-hand side, being a finite sum of normally distributed random variable, is itself normally distributed, with mean and covariance given by:

$$\mathbb{E}[I_n] = \left(\frac{b-a}{n}\right) \sum_{i=0}^{n-1} \mu(t_i^n), \qquad \mathbb{E}|I_n - \mathbb{E}[I_n]|^2 = \left|\frac{b-a}{n}\right|^2 \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} C(t_i^n, t_j^n).$$

Since μ and C are uniformly continuous, it is clear that

$$\mathbb{E}[I_n] \to \int_a^b \mu(t) \, \mathrm{d}t \quad \text{and} \quad \mathbb{E}|I_n - \mathbb{E}[I_n]|^2 \to \int_a^b \int_a^b C(s,t) \, \mathrm{d}s \, \mathrm{d}t \qquad \text{as } n \to \infty,$$

and so (this is easy to check using one of the equivalent definitions of convergence in distribution given by the Portmanteau lemma)

$$I_n \xrightarrow{d} \mathcal{N}\left(\int_a^b \mu(t) \, \mathrm{d}t, \int_a^b \int_a^b C(s,t) \, \mathrm{d}s \, \mathrm{d}t\right).$$

From (4), we also know that I_n converges to I almost surely and therefore also in distribution, which leads to the conclusion by invoking the uniqueness of the limit in distribution. □

The fact that the coefficients Z_i of the Karhunen–Loève expansion are pairwise independent for Gaussian processes (because uncorrelated Gaussians are also indepedent) means that we can employ the expansion to simulate Gaussian processes, as is done in the Jupyter notebook and in the problem sheet. The Karhunen–Loève expansion also enjoys the nice property that it minimizes the total mean-square error:

Proposition 11. *Let X_t be a zero-mean process with continuous autocovariance function $C(\cdot,\cdot)$, and let $\{\psi_n\}_{n\in\mathbf{N}}$ be a complete orthonormal basis of $L^2(a,b)$. For any basis, we saw that the process X_t may be approximated as*

$$X_t \approx S_n(t) := \sum_{i=1}^{n} A_i \, \psi_i(t), \qquad A_i := \int_a^b X_t \, \psi_i(t) \, \mathrm{d}t,$$

and that the total mean-square error of the approximation tends to 0 as $n \to \infty$. We will now show that, of all such approximations, the Karhunen–Loève expansion is the one that minimizes the total mean square error, provided that the eigenvalue are arranged in decreasing order.

Proof. The mean-square error, which we denote by ε_n, admits the expression:

$$\varepsilon_n^2 = \mathbb{E}\left[\int_a^b |X_t - S_n(t)|^2 \, \mathrm{d}t\right] = \mathbb{E}\left[\int_a^b \left|\sum_{i=n+1}^{\infty} A_i \, \psi_i(t)\right|^2 \mathrm{d}t\right].$$

Employing the continuity of the inner product (or of the norm) on $L^2(a,b)$ and the orthonormality

of the basis $\{\psi_i\}_{i\in\mathbf{N}}$, we obtain

$$\varepsilon_n^2 = \mathbb{E}\left[\sum_{i=n+1}^{\infty} |A_i|^2\right] = \sum_{i=n+1}^{\infty} \mathbb{E}|A_i|^2.$$

To avoid infinite series, we will show that the Karhunen–Loève expansion maximizes

$$J_*(\psi_1,\ldots,\psi_n) = \mathbb{E}\left[\int_a^b X_t^2\,\mathrm{d}t\right] - \varepsilon_n^2 = \sum_{i=1}^{n} \mathbb{E}|A_i|^2.$$

Employing Fubini–Tonelli's theorem, we deduce

$$J_*(\psi_1,\ldots,\psi_n) = \sum_{i=1}^{n} \int_a^b \int_a^b C(s,t)\,\psi_i(s)\,\psi_i(t)\,\mathrm{d}s\,\mathrm{d}t.$$

A necessary condition for an orthonormal basis $(\psi_1,\ldots,\psi_n)$ to maximize this functional is that all the functional derivatives of the Lagrangian

$$J(\psi_1,\ldots,\psi_n) = \sum_{i=1}^{n} \left(\int_a^b \int_a^b C(s,t)\,\psi_i(s)\,\psi_i(t)\,\mathrm{d}s\,\mathrm{d}t - \beta_i\left(\int_a^b \psi_i(t)^2\,\mathrm{d}t - 1\right)\right),$$

formed from the constraints that $\{\psi_i\}_{i\in\mathbf{N}}$ have norm 1 in $L^2(a,b)$, are zero at $(\psi_1,\ldots,\psi_n)$. A simple calculation shows that the functional derivatives (which are themselves functions on $[a,b]$) are given by:

$$\frac{\delta J}{\delta\psi_i}(\psi_1,\ldots,\psi_n) = 2\left(\int_a^b C(s,t)\,\psi_i(s)\,\mathrm{d}s - \beta_i\,\psi_i(t)\right), \qquad i = 1,\ldots,n.$$

For these to be zero, we deduce that $\{\psi_i\}_{i=1}^n$ must be eigenfunctions of the operator K_C. If this is the case, and denoting by λ_i the corresponding eigenvalues, then

$$J_*(\psi_1,\ldots,\psi_n) = \sum_{i=1}^{n} \lambda_i,$$

which is maximized when the eigenpairs are arranged in such a way that the eigenvalues are nonincreasing. □

3.3. Karhunen-Loève expansion

Another useful way to simulate stochastic processes is to use its Karhunen-Loève expansion. This states that every centered stochastic process with continuous covariance can be expanded into a random Fourier Series.

Note that:

- We can always write an arbitrary process Y_t as $Y_t = \mathbb{E}(Y_t) + X_t$ where $X_t = Y_t - \mathbb{E}(Y_t)$ is a centered process and consider the KL expansion of X_t.
- A stochastic process with continuous autocorrelation function (covariance) is mean-square continuous, i.e., $\lim_{\varepsilon \to 0} \mathbb{E}\left((X_{t+\varepsilon} - X_t)^2\right) = 0$.

Let X_t be a centered stochastic process with covariance $C(t,s)$. We define the operator $K : L^2(T) \longrightarrow L^2(T)$ by

$$K f(t) = \int_0^1 C(s,t) f(s)\, ds$$

(for simplicity, we let $T = [0,1]$).

Since the kernel of the operator K is a continuous, symmetric and non negative definite function, its eigenfunctions are orthogonal and form a complete set. The eigenvalues and eigenfunctions of K solve the integral equation

$$\int_0^1 C(t,s)\, \phi_n(s)\, ds = \lambda_n \phi_n(t).$$

and we can write $C(t,s)$ as

$$C(t,s) = \sum_{n=0}^{\infty} \lambda_n \phi_n(t) \phi_n(s).$$

Note that since the basis functions are orthogonal, they can be normalised such that

$$\int_0^1 \phi_n(t) \phi_m(t) \, dt = \delta_{mn}$$

We can use the basis $\{\phi_n\}$ of $L^2(T)$ to expand the process X_t:

$$X(t,\omega) = \sum_{n=0}^{\infty} \xi_n(\omega) \sqrt{\lambda_n} \phi_n(t)$$

where the ξ_n's are random variables with $\mathbb{E}(\xi_n) = 0$, $\mathbb{E}(\xi_n \xi_m) = \delta_{nm}$.

- this series is convergent in $L^2(\mathbb{P})$ for every $t \in T$.
- the functions / random variables ξ_n are computed in the same way as a Fourier series:

$$\xi_n \sqrt{\lambda_n} = \int_0^1 X_t \phi_n(t) \, dt.$$

Note that: the r.v. $\{\xi_n\}$ are orthogonal $\Rightarrow$ uncorrelated but not necessarily independent

We can check that

$$\mathbb{E}(\xi_n) = \mathbb{E}\left(\int_0^1 X_t \phi_n(t)\right) dt = \int_0^1 \mathbb{E}(X_t) \phi_n(t) \, dt$$

$$= 0 !$$

and

$$\mathbb{E}(\xi_n \xi_m) = \mathbb{E}\left(\int_0^1 \int_0^1 X_t X_s \phi_n(t) \phi_m(s)\, dt\, ds\right)$$

$$= \int_0^1 \int_0^1 \mathbb{E}(X_t X_s) \phi_n(t) \phi_m(s)\, dt\, ds$$

$$= \int_0^1 \int_0^1 C(t,s) \phi_n(t) \phi_m(s)\, dt\, ds$$

$$= \int_0^1 \lambda_n \phi_n(s) \phi_m(s)\, ds = \delta_{mn} \lambda_n.$$

The convergence in L^2 follows from Mercer's theorem (no need to know).

Lemma: The KL expansion minimises the mean squared error.

Proof: Consider any expansion $X_t = \sum_{n=0}^{\infty} \xi_n(\omega) \varphi_n(t)$ with φ_n an orthonormal basis, and its truncated version $X_t^{(M)} = \sum_{n=0}^{M} \xi_n(\omega) \varphi_n(t)$

The mean squared error is $MSE = \mathbb{E}(\varepsilon_M^2)$ where

$$\varepsilon_M = \mathbb{E}\left(X_t - X_t^{(M)}\right) = \mathbb{E}\left(\sum_{n=M+1}^{\infty} \xi_n(\omega) \varphi_n(t)\right)$$

Using the fact that $\xi_n = \int_0^1 X_t \varphi_n(t)\, dt$, we have

$$\varepsilon_M^2 = \sum_{n=M+1}^{\infty} \sum_{m=M+1}^{\infty} \varphi_n(t) \varphi_m(t) \int_0^1 \int_0^1 \mathbb{E}(X_t X_s) \varphi_n(t) \varphi_m(s)\, dt\, ds$$

$$= \sum_{n=M+1}^{\infty} \sum_{m=M+1}^{\infty} \varphi_n(t) \varphi_m(t) \int_0^1 \int_0^1 C(t,s) \varphi_n(t) \varphi_m(s)\, dt\, ds$$

Now we can integrate and use orthonormality of the basis $\{\varphi_n\}$ to obtain

$$\int_0^1 \varepsilon_t^2 \, dt = \sum_{n=k+1}^{\infty} \int_0^1 \int_0^1 C(t,s) \, \varphi_n(t) \, \varphi_n(s) \, dt \, ds$$

If we minimise this subject to orthonormality of the basis $\{\varphi_n\}$, we obtain that $\varphi_n = \phi_n$, the eigenfunctions of $C(t,s)$.

- The KL expansion is unique, i.e., it is the only expansion such that the r.v.'s $\{\xi_n\}$ are orthonormal!

- In contrast with Fourier series, where the basis functions are <u>always</u> $\{\sin(nt), \cos(nt)\}$, here the basis functions depend on the process X_t, more precisely on its covariance matrix $C(t,s)$. One can say that it adapts to the process in order to produce the best possible basis for the expansion.

- If X_t is a Gaussian stochastic process, then the ξ_n are Gaussian random variables
  $\Rightarrow$ they are independent Gaussian random variables.

<u>example: KL expansion for Brownian motion</u>

If $T=[0,1]$ and $X_t = W_t$ is a Brownian motion, we have

$$C(t,s) = \min(t,s)$$

The eigenvalue problem becomes

$$\int_0^1 \min(t,s) \, \phi_n(s) \, ds = \lambda_n \, \phi_n(t)$$

Or, computing C_1,

$$\int_0^t s\,\phi_n(s)\,ds + \int_t^1 t\,\phi_n(s)\,ds = \lambda_n \phi_n(t).$$

We can differentiate this equation twice:

$$t\,\phi_n(t) + \int_t^1 \phi_n(s)\,ds - t\,\phi_n(t) = \lambda_n \phi_n'(t)$$

$$\Leftrightarrow \int_t^1 \phi_n(s)\,ds = \lambda_n \phi_n'(t)$$

and, (differentiating again)

$$-\phi_n(t) = \lambda_n \phi_n''(t)$$

So the eigenvalues and eigenfunctions solve the differential equation

$$\phi_n''(t) = -\frac{1}{\lambda_n}\phi_n(t) \qquad (*)$$

with the boundary conditions $\phi_n(0) = \phi_n'(1) = 0$ (which we needed to obtain the above equations).

The solutions to (*) with the above boundary conditions are

$$\phi_n(t) = \sqrt{2}\sin\left(\frac{1}{2}(2n-1)\pi t\right)$$

$$\lambda_n = \left(\frac{2}{(2n-1)\pi}\right)^2$$

and therefore we can write

$$W_t = \sqrt{2}\sum_{n=0}^{\infty} \xi_n \frac{\sin\left(\left(n-\frac{1}{2}\right)\pi t\right)}{\left(n-\frac{1}{2}\right)\pi}$$

with $\xi_n = \int_0^1 W_t \sin\left(\left(n-\tfrac{1}{2}\right)\pi t\right)\sqrt{2}\,dt$.

($\xi_n \sim N(0,1)$ i.i.d.)

We can compute the eigenvalues and eigenvectors of the Brownian Bridge $B_t = W_t - tW$ similarly (exercise) and obtain

$$B_t = \sum_{n=1}^{\infty} \xi_n \sqrt{2} \frac{\sin(n\pi t)}{n\pi} .$$

Algorithm to generate paths of a centered stochastic process X_t:

(1) Solve the eigenvalue problem

$$\int C(t,s) \varphi_n(s)\, ds = \lambda_n \varphi_n(t)$$

(2) Calculate $\xi_n = \int X_t \varphi_n(t)\, dt$

(3) Use the truncated series

$$X_t^M = \sum_{m=1}^{M} \xi_m \varphi_m(t) .$$

Note that: the choice of the truncation M is important. We can conclude, from spectral analysis results, that the faster the autocorrelation function tends to zero, the broader is the corresponding spectral density, and therefore the greater is the number of required terms in the KL expansion.

3.3.1. Simulation of Gaussian Random Fields.

So far, we have looked at stochastic processes, which are random variables parametrised by time $t \in [0,T]$. However, we can consider r.v.'s parametrised by $x \in D$, $D \subset \mathbb{R}^d$.

Definition: Let D be a subset of $\mathbb{R}^d$. A random field $Z(\underline{x}, \omega)$ is a set of real valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$: for each $\underline{x} \in D$, $Z(\underline{x}, \cdot) : \Omega \to \mathbb{R}$ is a random variable.

- As before, for a fixed $\omega \in \Omega$, the associated realisation of a random field $Z(\underline{x}, \omega)$ is a deterministic function $f : D \to \mathbb{R}$. $f(\underline{x}) = Z(\underline{x}, \omega)$.

- Similarly, a Gaussian Random field is a Random field whose finite dimensional distribution is Gaussian.

Definition: We say that $Z(\underline{x}, \omega)$ is a second order random field if

$$\mathbb{E}(Z(\underline{x}, \omega)^2) = \int_\Omega |Z(x, \omega)|^2 \, d\mathbb{P}(\omega) < +\infty.$$

If $Z(\underline{x}, \omega)$ is a second order random field, we can define its mean and covariance:

$$\mu(\underline{x}) = \mathbb{E}(Z(\underline{x}, \omega))$$

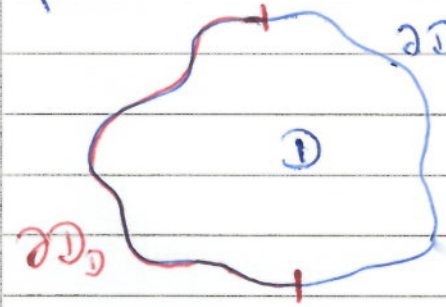
$$C(\underline{x}, \underline{y}) = \mathbb{E}\left((Z(\underline{x}, \omega) - \mu(\underline{x}))(Z(\underline{y}, \omega) - \mu(\underline{y}))\right).$$

for $\underline{x}, \underline{y} \in D$

Note that: The smoothness / regularity of $Z(\underline{x}, \omega)$ depends on the regularity of $C(\underline{x}, \underline{y})$.

- if C is continuous, then $Z(\underline{x}, \omega)$ is mean-square continuous.
- if $C \in C^2(D \times D)$, then $Z(\underline{x}, \omega)$ is differentiable.

example of application: Stationary flow in a porous medium



$$-\nabla \cdot (A(\underline{x}, \omega) \nabla p) = f \quad \text{in } D$$
$$p = g \quad \text{on } \partial D_D$$
$$\underline{u} \cdot \vec{n} = 0 \quad \text{on } \partial D_N$$

$p \rightarrow$ pressure field

$\underline{u} = A(\underline{x}, \omega) \cdot \nabla p \rightarrow$ velocity field.

$A(\underline{x}, \omega) \rightarrow$ permeability tensor.

A very nice application is to solve the inverse problem: identify $A(\underline{x}, \omega)$ from observations (for example, in the boundary)

On the other hand, we might want to solve the direct problem. consider

$$A(\underline{x}, \omega) = \exp(Z(\underline{x}, \omega))$$

and calculate the statistics of the pressure and velocity fields $p(\underline{x}, \omega)$, $u(\underline{x}, \omega)$

$\Rightarrow$ infinitely many PDE solves $\Rightarrow$ need to generate realisations of $A(\underline{x}, \omega)$ in an efficient way $\Rightarrow$ KL expansion.

examples of Random fields:

① Separable exponential covariance function

$$C(\underline{x},\underline{y}) = \sigma^2 \prod_{i=1}^{d} \exp\left(-\frac{|x_i - y_i|}{\ell_i}\right)$$

$\sigma^2 \to$ variance $(C(\underline{x},\underline{x}) = \sigma^2)$

ℓ_i, $i = 1, \dots, d \to$ correlation length, $\ell_i > 0$.

Note that: can obtain KL expansion as a product of 1D variables.

② Exponential covariance function

$$C(\underline{x},\underline{y}) = \sigma^2 \exp\left(-\frac{\|x - y\|_2}{\ell}\right), \quad \ell > 0$$

$\|x_2\| = \sqrt{\sum_{i=1}^{d} x_i^2}$. ℓ = correlation length as before.

Note that: KL is harder here. Also, the paths here are not smooth

③ Gaussian covariance function

$$C(\underline{x},\underline{y}) = \sigma^2 \exp\left(-\frac{\|\underline{x} - \underline{y}\|_2^2}{\ell}\right), \quad \ell > 0.$$

→ generates smooth paths.

3.3.2. Solution of the integral equation

The bottleneck of obtaining the KL expansion of a stochastic process of random field is in the ability of solving the integral equation

$$\int_D C(x,y)\,\phi(y)\,dy = \lambda\,\phi(x).$$

In certain cases, we can solve this equation analytically. For example, if X_t (or $Z(x,\omega)$) is a stationary process* (or random field) then we can write

$$C(x,y) = \mathcal{C}(x-y).$$

and $\mathcal{C}$ has a rational spectral density:

$$S(\omega) = \frac{1}{2\pi}\int e^{-ix\omega}\,C(x)\,dx \text{ is}$$

such that $S(\omega) = \frac{N(\omega^2)}{D(\omega^2)} = \frac{\text{polynomial of deg } n}{\text{polynomial of deg. } d}$

then the integral equation becomes a differential equation

$$N\left(\frac{d^2}{dx^2}\right)\phi_n(x) = \lambda_n D\left(\frac{d^2}{dx^2}\right)\phi_n(x)$$

which we can solve. Note that this is exactly what we did for the Brownian Motion example! (even though the BM is NOT a stationary process!)

* We will discuss stationary processes in Section 5.

In other cases, it is not possible to solve the integral equation analytically and we must resort to its numerical solution.

① Collocation Method

Select points $x_1, x_2, \dots, x_N \in \mathcal{D}$ and compute

$$\int_{\mathcal{D}} C(x_k, y)\, \phi_n(y)\, dy = \lambda_n\, \phi_n(x_k)$$

for $k = 1, 2 \dots, N$.

We must use numerical integration techniques:

$$\sum_{i=1}^{N} w_i\, C(x_k, x_i)\, \phi_n(x_i) = \lambda_n\, \phi_n(x_k)$$

(w_i: weight from numerical integration)

$k = 1, \dots, N$. This allows us to obtain a <u>generalised eigenvalue problem</u>

$$C W \underline{\Phi} = \Lambda \underline{\Phi} \qquad (*)$$

where $W \in \mathbb{R}^{N \times N}$ is a matrix with the quadrature weights, Λ a diagonal matrix of e-values, Φ a matrix of eigenfunctions and $C_{ij} = C(x_i, x_j)$.

The matrix CW is dense and the solution of $(*)$ can be computationally expensive

② Galerkin Method

An alternative way is to consider a basis $\{h_i(x)\}_{i=1}^{\infty}$ for the Hilbert space $H = L^2(D)$ and expand the eigenfunctions in this basis

$$\phi_n^{(N)}(x) = \sum_{\ell=1}^{\infty} d_\ell^{(N)} h_\ell(x)$$

Plugging this into the integral equation, we obtain the error:

$$\varepsilon_N = \sum_{\ell=1} d_\ell^{(N)} \left(\int C(x,y) h_\ell(y) dy - \lambda_n h_\ell(x) \right)$$

We can then require the error to be orthogonal (in $L^2(D)$) to the basis functions (like in the finite elements setting).

$$\varepsilon_N \perp h_i, \quad i = 1, \dots, N \Rightarrow$$

$$\Rightarrow \int_D \varepsilon_N(x) h_i(x) dx = 0, \quad i = 1, \dots, N$$

and thus gives rise to another generalised eigenvalue problem

$$CD = \Lambda B D,$$

where

$$C_{ij} = \iint C(x,y) h_i(y) h_j(x) dx dy$$

$$B_{ij} = \int_D h_i(x) h_j(x) dx$$

$$D_{ij} = d_i^{(j)}, \quad \Lambda_{ij} = \delta_{ij} \lambda_j.$$

We can use polynomials as the basis.

$\Rightarrow$ this method is more attractive.

It can be shown to be equivalent to a variational treatment of the problem.