List of examinable proofs

This list gathers the results shown in class of which the proof is examinable. It will grow as the course progresses and might be subject to modifications.

Monte Carlo simulation

- The inverse method works (Lemma 2.1).
- Rejection sampling works (Lemma 2.3).
- The Box–Muller algorithm works (Exercise 2.6).
- Derive an expression for the variance of the Monte Carlo estimator.
- The Monte Carlo estimator is unbiased and strongly consistent (Proposition 2.7).
- The Monte Carlo estimator is asymptotically normal (Proposition 2.8).
- Variance reduction and optimal α with control variate (p22 and Example 2.5).
- Variance of Monte Carlo estimator with importance sampling (Proposition 2.10).
- Minimizer and minimum of the variance with importance sampling (Proposition 2.11).
- Conditioning leads to a variance reduction (Lemma 2.9 and following calculation).

Continuous-time Gaussian processes

- A Gaussian process is weakly stationary if and only if it is strongly stationary (Lemma 4.3).
- Mean-square ergodic theorem for weakly stationary processes – the assumptions of Tonelli–Fubini need not be checked explicitly (Theorem 6 in the notes of w3).
- Karhunen–Loève theorem – you should be able to prove all the statements when provided with the statement of Mercer's theorem (Theorem 10 in the notes of w3).
- Feynman–Kac formula for the solution to the *backward Kolmogorov equation* (Theorem 2 in the weekly notes of w6).
- Derive the Milstein's scheme for stochastic differential equations.

Inference for stochastic differential equations

- Show that $|\mathbb{E}[\hat{\sigma}_N^2 - \sigma^2]| \leq C\sqrt{\Delta t}$ (Proposition 2 in the notes of w7).
- Write down the probability density function of the discrete-time process $\{X_n\}_{n=1}^N$ obtained from the following iteration,

$$X_{n+1} = X_n - b(X_n; \boldsymbol{\theta})\,\Delta t + \sqrt{\Delta t}\,\xi_n, \qquad n = 0, \dots, N-1, \qquad X_0 = x_0, \tag{1}$$

where $\boldsymbol{\theta}$ is a vector of parameters. Given a precise expression of $b(X_n, \boldsymbol{\theta})$, for example $b(X_n; \boldsymbol{\theta}) = -\theta\, b(X_n)$, obtain the maximum likelihood estimator for the parameters based on one or several realizations of (1).

- Obtain the maximum likelihood estimator for the parameters $\boldsymbol{\theta}$ in the stochastic differential equation
$$\mathrm{d}X_t = -b(X_t; \boldsymbol{\theta})\,\mathrm{d}t + \mathrm{d}W_t, \qquad t \in [0, T], \qquad X_0 = x_0, \tag{2}$$
based on one or several paths of the solution to (2).

Markov chain Monte Carlo

- Definition of the stationary distribution of a Markov chain (Definition 3.3).
- If a probability distribution satisfies the detailed balance condition, then it is a stationary distribution of the Markov chain (Theorem 3.2).
- Derive the transition probability of the Metropolis–Hastings algorithm, i.e. obtain equation 3.8 in the lecture notes from 2016.
- The density π is reversible with respect to the transition probability associated with the Metropolis–Hastings algorithm (Proposition 3.5).
- Derive a formula for the mean and variance of $I_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$ when Z_i is a stationary Markov chain.

Note that...

... The exam will not be only about reproducing proofs. You are expected to also be able to solve exercises of the type discussed in the course, for example (but this is not an exhaustive list)

- Apply the Kolmogorov–Smirnov test.
- Describe and apply the methods for generating non-uniform random variables.
- Calculate the acceptance probability of a rejection sampling algorithm.
- Construct a confidence interval for a Monte Carlo estimator, using either Chebychev's theorem, the central limit theorem, or Bikelis' theorem.
- Find control variates or importance distributions that result in a variance reduction for Monte Carlo estimators.
- Describe an algorithm to simulate Gaussian processes on a mesh by "direct simulation".
- Obtain an update formula to generate Markovian Gaussian processes iteratively (Jupyter notebook of w3).
- Calculate the KL expansion for simple processes, such as Brownian motion and the Brownian bridge (The procedure is outlined in the notes and the Jupyter notebook of w3).

- Show that the weak error associated with simple numerical schemes converges to zero in the limit as $N \to \infty$.
- Study the mean-square and asymptotic stability of a given numerical scheme for SDEs.
- Prove the convergence to zero of the weak error of Euler–Maruyama for the observable $f(x) = x$, in the specific case of geometric Brownian motion.
- Calculate the mean-square error of simple estimators.
- Apply Lamperti's transformation to obtain a stochastic differential equation with diffusion coefficient 1.
- Given an algorithm for generating Markov chains, obtain the transition probability and check (either directly or via detailed balance) whether a given distribution π is a stationary distribution of the Markov chain.