M4A44 Mock Exam

April 16, 2016

Question 1

1. Given $\alpha > 0$ and $\lambda > 0$, consider the truncated exponential distribution

$$g_{\alpha,\lambda}(x) = \begin{cases} \frac{1}{Z} e^{-\lambda x} & 0 \leq x \leq \alpha. \\ 0 & \text{otherwise} \end{cases}, \tag{1}$$

where Z is a normalization constant.

   (i) Compute the cumulative distribution function of $g_{\alpha,\lambda}$ and its inverse.

   (ii) Using the inverse transform method, construct a sampler which generates IID samples of $g_{\alpha,\lambda}$ given a stream of IID $U(0,1)$ distribution random numbers.

2. Suppose we wish to sample from the distribution with density

$$f(x) = 32\frac{x(1-x)e^{-4(x-1)}}{3+e^4}$$

supported on $[0,1]$.

   (i) We wish to implement a rejection sampler for $f(x)$ using a proposal density $h(x) = 1$, on $[0,1]$. Show that

$$x_1 = \arg\max_{x\in[0,1]} \frac{f(x)}{h(x)} = \frac{1}{4}(3-\sqrt{5}).$$

   (ii) Write down the steps of a rejection sampler for $f(x)$ using $h(x) = 1$.

   (iii) Let the random variable Z be the output of the rejection sampler. Show that Z is distributed according to the density f.

   (iv) Suppose instead we use proposal density $g_{1,4}$, where $g_{\alpha,\lambda}$ is given by (1). Show that

$$x_1 = \arg\max_{x\in[0,1]} \frac{f(x)}{g_{1,4}(x)} = \frac{1}{2}$$

   (v) Given that $f(x_1)/h(x_1) > f(x_2)/g_{1,4}(x_2)$, what does that say about the relative performance of both rejection samplers?

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Question 2

Suppose we wish to sample from the distribution with smooth positive density $\pi(x)$ on $\mathbb{R}$ using MCMC.

1. We want to sample from $\pi(x)$ using a Metropolis-Hastings algorithms with proposal
$$y = \sqrt{1-\beta^2}x + \beta w, \quad w \sim \mathcal{N}(0,1),$$
where $\beta \in [0,1]$ is a constant.
   - (i) Describe the steps of a Metropolis-Hastings algorithm using the proposal $q(y \mid x)$ to sample from π, giving expressions for the proposal density and acceptance probability.
   - (ii) Given a function $f \in L^1(\pi)$, describe how to use the above Metropolis-Hastings algorithm to define an estimator $\hat{I}_n$ for $I = \mathbb{E}_\pi[f]$, where $n \in \mathbb{N}$.
   - (iii) What does it mean for an estimator $\hat{I}_n$ to be (a) unbiased and (b) consistent?
   - (iv) Show that the estimator $\hat{I}_n$ defined above is consistent. Is it unbiased?

2. Consider an MCMC scheme for sampling from π which given $X_n = x$, a new state y is proposed according to
$$y = x + \sqrt{\delta}w, \quad w \sim \mathcal{N}(0,1),$$
where $\delta > 0$, and is accepted with probability
$$\alpha(x,y) = \frac{\pi(y)}{\pi(x) + \pi(y)}.$$
   - (i) Write down the probability transition function $p(x,y) = \mathbb{P}\left[X_{n+1} = y \,|\, X_n = x\right]$ for the Markov chain defined above.
   - (ii) What does it mean for π to be reversible with respect to $p(x,y)$?
   - (iii) Prove that π is reversible with respect to $p(x,y)$.

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Question 3

1. Give the definition of a continuous time Gaussian process.
2. Give the definition for a continuous time process to be strictly stationary and weakly stationary.
3. Let $X(t)$ be a Gaussian process with mean $\mu(t)$ and covavariance $C(s,t)$, where $0 \leq s,t \leq T$. Write down an algorithm to generate the sample $(X(t_1), X(t_2), \ldots, X(t_N))$, for $0 \leq t_1 < t_2 < \ldots < t_N \leq T$.

4. Suppose we to generate $X(t_{n+1})$ given that we have already generate $X(t_0), \ldots, X(t_n)$.

   (i) Consider
   $$\mathbf{m} = (m_1, m_2) \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix},$$
   where Σ is positive definite. Let $\mathbf{X} = (X_1, X_2) \sim \mathcal{N}(\mathbf{m}, \Sigma)$. Show that the conditional distribution of X_2 conditional on X_1 is a multivariate normal with
   $$\mathbb{E}[X_2 \,|\, X_1] = m_2 + \frac{\sigma_{21}}{\sigma_{11}}(X_1 - m_1).$$
   and
   $$\mathrm{Var}(X_2 \,|\, X_1) = \sigma_{22} - \frac{\sigma_{21}}{\sigma_{11}}\sigma_{12}.$$

   (ii) Suppose that $X(t)$ is a Gaussian Markov process. Derive a scheme to generate samples of $X(t_{n+1})$ given values $X(t_n), \ldots, X(t_0)$, where $0 \leq t_0 < t_1 < \ldots < t_n < t_{n+1}$.

   (iii) Apply this method to sample from a Gaussian process mean 0 and covariance $C(s,t) = \exp(-\alpha|t-s|/2)$, and show that
   $$X(t_{n+1}) = e^{-\alpha|t_n - t_{n+1}|/2} X(t_n) + \sqrt{1 - e^{-\alpha|t_n - t_{n+1}|}}\, w,$$
   where $w \sim \mathcal{N}(0,1)$.

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Question 4

Consider the following scalar Itô SDE
$$dX_t = b(X_t)\,dt + \sqrt{(}X_t), \tag{2}$$
where W_t is a standard one-dimensional Brownian motion and $b(x)$ and $\sigma(x)$ are smooth, bounded functions.

1. In this question we consider the Euler-Maruyama discretisation for (2).

   (i) Derive the Euler-Maruyama discretisation for (2) for approximating $X(t_n)$, where $t_n = n\Delta t$, $n \in \mathbb{N}$.

   (ii) Define the strong and weak orders of convergence of a numerical approximation to (2).

   (iii) What are the strong and weak orders of convergence of the Euler-Maruyama scheme?

2. In this question we derive the Milstein approximation of (2).

(i) Using the fact that
$$dW_t^2 = 2W_t\, dW_t + dt,$$
show that
$$\int_{n\Delta t}^{(n+1)\Delta t} W_t\, dW_t = \frac{1}{2}(\Delta W_n^2 - \Delta t),$$
where $\Delta W_n = W_{(n+1)\Delta t} - W_{n\Delta t}$.

(ii) Derive the Milstein approximation of (2).

(iii) What are the strong and weak orders of convegence of the Milstein scheme?

3. Given $\theta \in [0, 1]$, consider the θ-Euler Maruyama approximation of (2) given by
$$X_{n+1} = X_n + [(1-\theta)b(X_n) + \theta b(X_{n+1})]\Delta t + \sigma(X_n)\Delta W_n.$$

(a) Consider the scalar Geometric Brownian motion
$$dX_t = \lambda X_t\, dt + \sigma X_t\, dW_t$$

(b) Obtain a formula for $\mathbb{E}|X_t|^2$, and show that the process X_t is mean square stable when
$$2\lambda + \sigma^2 < 0.$$

(c) Write down the θ-Euler Maruyama discretisation X_n of the scalar Geometric Brownian motion process.

(d) Show that X_n is mean-square stable when
$$2\lambda + \sigma^2 + \Delta t(1-2\theta)\lambda^2 < 0.$$

(e) What happens when $\theta = 1/2$?

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