Appendix C

Chebyshev polynomials

The Chebyshev polynomials $(T_n)_{n\in\mathbf{N}}$ are given on $[-1,1]$ by the formula

$$\forall x \in [-1,1], \qquad T_n(x) = \cos\bigl(n \arccos(x)\bigr). \tag{C.1}$$

Although this formula makes sense only if $x \in [-1,1]$, the polynomials are defined for all $x \in \mathbf{R}$. Equivalently, the Chebyshev polynomials can be defined from the equation

$$\forall x \in [1,\infty), \qquad T_n(x) = \cosh\bigl(n \operatorname{arccosh}(x)\bigr), \tag{C.2}$$

where $\cosh(\theta) = \frac{1}{2}\left(\mathrm{e}^{\theta} + \mathrm{e}^{-\theta}\right)$ and $\operatorname{arccosh}\colon [1,\infty) \to [0,\infty)$ is the inverse function of cosh. The first few Chebyshev polynomials are illustrated in Figure C.1. It is immediate to show the following properties from (C.1):

- The roots of T_n are given by

$$z_k = \cos\left(\frac{\pi}{2n} + \frac{k\pi}{n}\right), \qquad k = 0,\dots,n-1.$$

  These are illustrated in Figure C.2.

- The polynomial T_n takes the value 1 or -1 when evaluated at

$$x_k = \cos\left(\frac{k\pi}{n}\right), \qquad k = 0,\dots,n. \tag{C.3}$$

  More precisely, it holds that $T_n(x_k) = (-1)^k$.

Exercise C.1. *Show that (C.1) defines a polynomial of degree n, and find its expression in the usual polynomial notation.*

Solution. The key idea is to rewrite the cosine function in terms of the complex exponential:

$$\cos(n\theta) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}n\theta} + \mathrm{e}^{-\mathrm{i}n\theta}\right) = \frac{1}{2}\Bigl(\bigl(\cos(\theta) + \mathrm{i}\sin(\theta)\bigr)^n + \bigl(\cos(\theta) - \mathrm{i}\sin(\theta)\bigr)^n\Bigr).$$

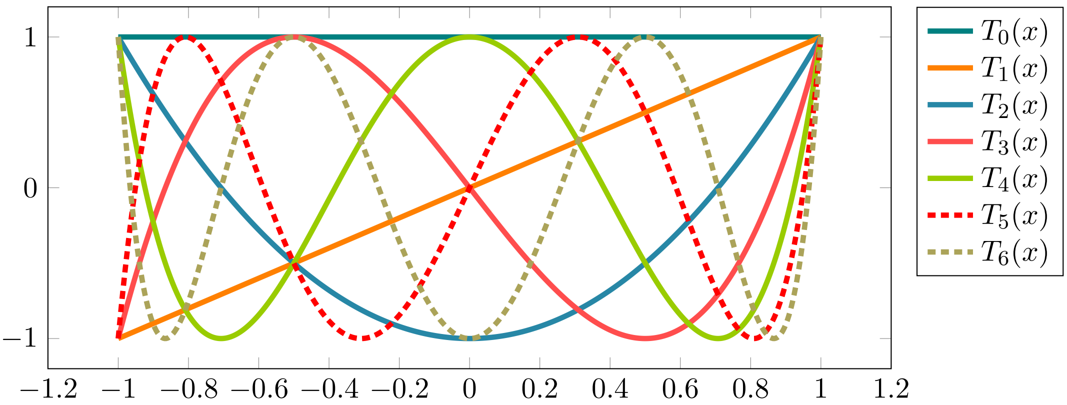

Figure C.1: Illustration of the first few Chebyshev polynomials over the interval $[-1, 1]$.

By expanding the powers on the right-hand side, we obtain

$$\begin{aligned}\bigl(\cos(\theta) + \mathrm{i}\sin(\theta)\bigr)^n &= \sum_{j=0}^{n} \binom{n}{j} \cos(\theta)^{n-j}\, \mathrm{i}^j \, \sin(\theta)^j \\ \bigl(\cos(\theta) - \mathrm{i}\sin(\theta)\bigr)^n &= \sum_{j=0}^{n} \binom{n}{j} \cos(\theta)^{n-j}\, (-\mathrm{i})^j \, \sin(\theta)^j.\end{aligned}$$

The terms corresponding to odd values of j cancel out in the expression of $\cos(n\theta)$, and so we obtain the following expression for $\cos(n\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$:

$$\begin{aligned}\cos(n\theta) &= \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \cos(\theta)^{n-2j} \mathrm{i}^{2j} \sin(\theta)^{2j} \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{2j} \cos(\theta)^{n-2j} \bigl(1 - \cos(\theta)^2\bigr)^j.\end{aligned}$$

Therefore, we conclude that

$$T_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} x^{n-2j} \bigl(x^2 - 1\bigr)^j. \tag{C.4}$$

△

Exercise C.2. *Show that the same polynomials are obtained from* (C.2).

Solution. Notice that

$$\begin{aligned}\cosh(n\xi) &= \frac{1}{2}\left(\mathrm{e}^{n\xi} + \mathrm{e}^{-n\xi}\right) \\ &= \frac{1}{2}\Bigl(\bigl(\cosh(\xi) + \sinh(\xi)\bigr)^n + \bigl(\cosh(\xi) - \sinh(\xi)\bigr)^n\Bigr).\end{aligned}$$

Using the binomial formula, we obtain

$$\begin{aligned}\cosh(n\xi) &= \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}\left(\cosh(\xi)^{n-j}\sinh(\xi)^j + \cosh(\xi)^{n-j}(-1)^j\sinh(\xi)^j\right)\\ &= \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}\cosh(\xi)^{n-j}\left(\sinh(\xi)^j + (-1)^j\sinh(\xi)^j\right).\end{aligned}$$

The contributions of the odd values of j cancel out, and so we obtain

$$\cosh(n\xi) = \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}\cosh(\xi)^{n-2j}\sinh(\xi)^{2j}.$$

Since $\cosh(\xi)^2 - \sinh(\xi)^2 = 1$, we deduce that

$$\cosh(n\xi) = \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{j}\cosh(\xi)^{n-2j}(\cosh(\xi)^2-1)^j,$$

which after the substitution of $\xi = \operatorname{arccosh}(x)$ leads to (C.4). △

Exercise C.3 (Yet another expression for the Chebyshev polynomials)**.** *Show that $T_n(x)$ may be defined from the formula*

$$T_n(x) = \frac{1}{2}\left(x+\sqrt{x^2-1}\right)^n + \frac{1}{2}\left(x-\sqrt{x^2-1}\right)^n \qquad \textit{for}\quad |x|\geq 1. \tag{C.5}$$

Solution. We showed in the solution of Exercise C.2 that

$$\cosh(n\xi) = \frac{1}{2}\Big(\big(\cosh(\xi)+\sinh(\xi)\big)^n + \big(\cosh(\xi)-\sinh(\xi)\big)^n\Big).$$

Letting $\xi = \operatorname{arccosh}(x)$ in this equation and using that $\cosh(\xi)^2 - \sinh(\xi)^2 = 1$, we obtain

$$T_n(x) = \frac{1}{2}\Big(\big(x+\sqrt{x^2-1}\big)^n + \big(x-\sqrt{x^2-1}\big)^n\Big),$$

which is the required formula. △

Exercise C.4 (Recursion relation)**.** *Show that the Chebyshev polynomials satisfy the relation*

$$\forall n\in\{1,2,\dots\},\qquad T_{n+1} = 2xT_n - T_{n-1}. \tag{C.6}$$

Solution. It is sufficient to show the identity for $x\in[-1,1]$, where the formula (C.1) applies. Using well-known trigonometric identities, we have

$$\begin{aligned}\cos\big((n+1)\theta\big) &= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta)\\ \cos\big((n-1)\theta\big) &= \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta).\end{aligned}$$

Adding both equations and rearranging, we obtain

$$\cos\big((n+1)\theta\big) = 2\cos(n\theta)\cos(\theta) - \cos\big((n-1)\theta\big).$$

Therefore, using this equation with $\theta = \arccos(x)$, we obtain the statement. △

Remark C.1. The recursion relation in Exercise C.4 can be employed to show by recursion that $T_n(x)$ is indeed a polynomial of degree n.

Exercise C.5. *Since $T_n : \mathbf{R} \to \mathbf{R}$ is a polynomial, it may be written in the standard form*

$$T_n(x) = \alpha_n^{(n)} x^n + \ldots + \alpha_1^{(n)} x + \alpha_0^{(n)}.$$

Prove that $\alpha_n^{(n)} = 2^{(n-1)}$ provided that $n \geqslant 1$.

Solution. From the definition (C.1), the Chebyshev polynomials of degrees 0 and 1 are given by $T_0(x) = 1$ and $T_1(x) = x$. The statement then follows by recursion, using Exercise C.4. △

Exercise C.6. *Let $\xi \in \mathbf{R} \backslash (-1, 1)$. Show that, among all the polynomials in $\mathbf{P}(n)$ that are bounded from above by 1 in absolute value uniformly over the interval $(-1, 1)$, the Chebyshev polynomial T_n achieves the largest absolute value when evaluated at ξ.*

Solution. Reasoning by contradiction, we assume that there exists $p \in \mathbf{P}(n)$ that satisfies

$$\sup_{x \in (-1,1)} |p(x)| \leqslant 1 \qquad \text{and} \qquad |p(\xi)| > |T_n(\xi)|.$$

Let $q(x) = p(x) T_n(\xi)/p(\xi)$. Then by construction $q(\xi) = T_n(\xi)$ and

$$\sup_{x \in (-1,1)} |q(x)| < 1.$$

Consequently, denoting by x_k the points defined in (C.3), we have that

$$\forall k \in \{0, \ldots, n\}, \qquad (-1)^k (T_n - q)(x_k) > 0.$$

In other words, the polynomial $T_n - q$ takes positive values at $\{x_0, x_2, x_4, \ldots\}$ and negative values at $\{x_1, x_3, x_5, \ldots\}$. Consequently, by the intermediate value theorem, $T_n - q$ possesses n distinct roots in the open interval $(-1, 1)$. Since, in addition, $(T_n - q)(\xi) = 0$, we deduce that $T_n - q$ has $n+1$ distinct roots, which is a contradiction given that $T_n - q$ is a nonzero polynomial of degree at most n. △

Exercise C.7. *Assume that $0 < \lambda_1 < \lambda_2$. Prove that for any polynomial $p \in \mathbf{P}(n)$ that satisfies $p(0) = 1$, it holds that*

$$\sup_{\lambda \in (\lambda_1, \lambda_2)} |p(\lambda)| \geqslant \frac{1}{T_n(\xi)}, \qquad \xi := \frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1},$$

with equality for

$$p_*(\lambda) = \frac{T_n\left(\frac{\lambda_1 + \lambda_2 - 2\lambda}{\lambda_2 - \lambda_1}\right)}{T_n\left(\frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1}\right)}. \tag{C.7}$$

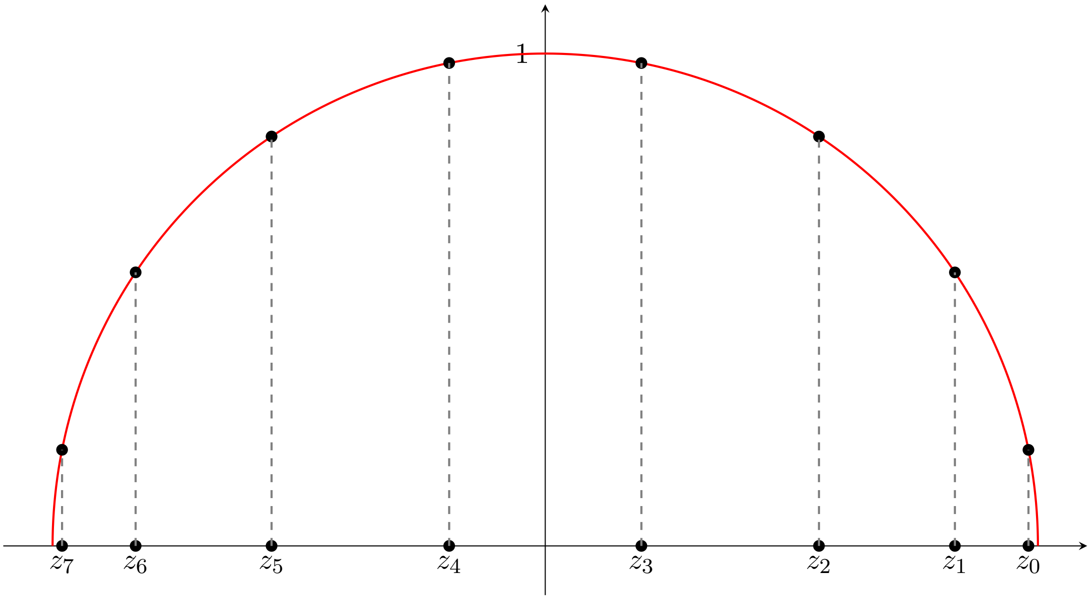

Figure C.2: Roots of the Chebyshev polynomial T_8.

Solution. Assume that $p \in \mathbf{P}(n)$ is such that $p(0) = 1$, and let $q \in \mathbf{P}(n)$ be given by

$$q(\mu) = p\left(\frac{\lambda_1 + \lambda_2 - (\lambda_2 - \lambda_1)\mu}{2}\right) \qquad \Leftrightarrow \qquad p(\lambda) = q\left(\frac{\lambda_1 + \lambda_2 - 2\lambda}{\lambda_2 - \lambda_1}\right).$$

Since $\xi > 1$, it holds from (C.5) that $T_n(\xi) > 0$ and it follows from Exercise C.6 that

$$p(0) = q(\xi) \leqslant T_n(\xi) \sup_{\mu \in (-1,1)} |q(\mu)| = T_n(\xi) \sup_{\lambda \in (\lambda_1, \lambda_2)} |p(\lambda)|,$$

with equality when $q \propto T_n$, i.e. when

$$p(\lambda) \propto T_n\left(\frac{\lambda_1 + \lambda_2 - 2\lambda}{\lambda_2 - \lambda_1}\right).$$

The expression (C.7) then follows from the fact that $p_*(0) = 1$. $\triangle$