

## Appendix C

# Chebyshev polynomials

The Chebyshev polynomials  $(T_n)_{n \in \mathbf{N}}$  are given on  $[-1, 1]$  by the formula

$$\forall x \in [-1, 1], \quad T_n(x) = \cos(n \arccos(x)). \quad (\text{C.1})$$

Although this formula makes sense only if  $x \in [-1, 1]$ , the polynomials are defined for all  $x \in \mathbf{R}$ . Equivalently, the Chebyshev polynomials can be defined from the equation

$$\forall x \in [1, \infty), \quad T_n(x) = \cosh(n \operatorname{arccosh}(x)), \quad (\text{C.2})$$

where  $\cosh(\theta) = \frac{1}{2}(e^\theta + e^{-\theta})$  and  $\operatorname{arccosh}: [1, \infty) \rightarrow [0, \infty)$  is the inverse function of  $\cosh$ . The first few Chebyshev polynomials are illustrated in [Figure C.1](#). It is immediate to show the following properties from (C.1):

- The roots of  $T_n$  are given by

$$z_k = \cos\left(\frac{\pi}{2n} + \frac{k\pi}{n}\right), \quad k = 0, \dots, n-1.$$

These are illustrated in [Figure C.2](#).

- The polynomial  $T_n$  takes the value 1 or -1 when evaluated at

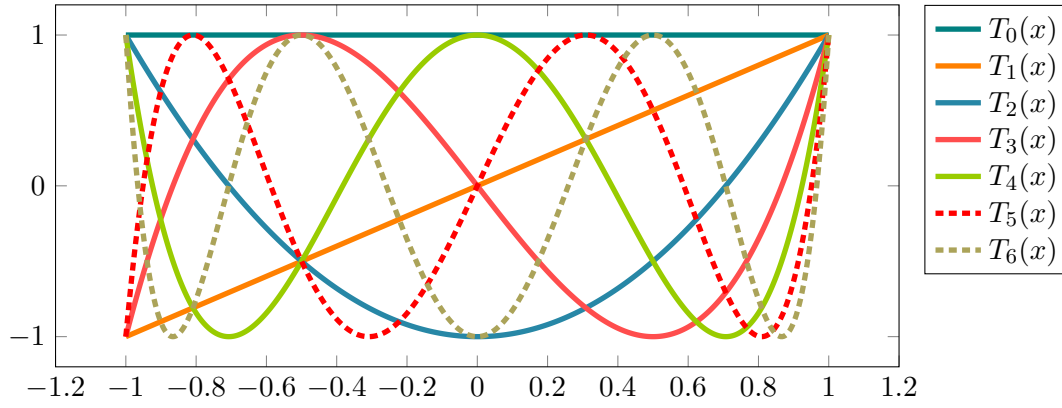
$$x_k = \cos\left(\frac{k\pi}{n}\right), \quad k = 0, \dots, n. \quad (\text{C.3})$$

More precisely, it holds that  $T_n(x_k) = (-1)^k$ .

**⚙️ Exercise C.1.** Show that (C.1) defines a polynomial of degree  $n$ , and find its expression in the usual polynomial notation.

*Solution.* The key idea is to rewrite the cosine function in terms of the complex exponential:

$$\cos(n\theta) = \frac{1}{2}(e^{in\theta} + e^{-in\theta}) = \frac{1}{2}\left((\cos(\theta) + i\sin(\theta))^n + (\cos(\theta) - i\sin(\theta))^n\right).$$


 Figure C.1: Illustration of the first few Chebyshev polynomials over the interval  $[-1, 1]$ .

By expanding the powers on the right-hand side, we obtain

$$\begin{aligned} (\cos(\theta) + i \sin(\theta))^n &= \sum_{j=0}^n \binom{n}{j} \cos(\theta)^{n-j} i^j \sin(\theta)^j \\ (\cos(\theta) - i \sin(\theta))^n &= \sum_{j=0}^n \binom{n}{j} \cos(\theta)^{n-j} (-i)^j \sin(\theta)^j. \end{aligned}$$

The terms corresponding to odd values of  $j$  cancel out in the expression of  $\cos(n\theta)$ , and so we obtain the following expression for  $\cos(n\theta)$  in terms of  $\cos(\theta)$  and  $\sin(\theta)$ :

$$\begin{aligned} \cos(n\theta) &= \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \cos(\theta)^{n-2j} i^{2j} \sin(\theta)^{2j} \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{2j} \cos(\theta)^{n-2j} (1 - \cos(\theta)^2)^j. \end{aligned}$$

Therefore, we conclude that

$$T_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} x^{n-2j} (x^2 - 1)^j. \quad (\text{C.4})$$

△

⚙️ **Exercise C.2.** Show that the same polynomials are obtained from (C.2).

*Solution.* Notice that

$$\begin{aligned} \cosh(n\xi) &= \frac{1}{2} (e^{n\xi} + e^{-n\xi}) \\ &= \frac{1}{2} \left( (\cosh(\xi) + \sinh(\xi))^n + (\cosh(\xi) - \sinh(\xi))^n \right). \end{aligned}$$

Using the binomial formula, we obtain

$$\begin{aligned} \cosh(n\xi) &= \frac{1}{2} \sum_{j=0}^n \binom{n}{j} (\cosh(\xi)^{n-j} \sinh(\xi)^j + \cosh(\xi)^{n-j} (-1)^j \sinh(\xi)^j) \\ &= \frac{1}{2} \sum_{j=0}^n \binom{n}{j} \cosh(\xi)^{n-j} (\sinh(\xi)^j + (-1)^j \sinh(\xi)^j). \end{aligned}$$

The contributions of the odd values of  $j$  cancel out, and so we obtain

$$\cosh(n\xi) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \cosh(\xi)^{n-2j} \sinh(\xi)^{2j}.$$

Since  $\cosh(\xi)^2 - \sinh(\xi)^2 = 1$ , we deduce that

$$\cosh(n\xi) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j} \cosh(\xi)^{n-2j} (\cosh(\xi)^2 - 1)^j,$$

which after the substitution of  $\xi = \operatorname{arccosh}(x)$  leads to (C.4).  $\triangle$

❁ **Exercise C.3** (Yet another expression for the Chebyshev polynomials). *Show that  $T_n(x)$  may be defined from the formula*

$$T_n(x) = \frac{1}{2} \left( x + \sqrt{x^2 - 1} \right)^n + \frac{1}{2} \left( x - \sqrt{x^2 - 1} \right)^n \quad \text{for } |x| \geq 1. \quad (\text{C.5})$$

*Solution.* We showed in the solution of Exercise C.2 that

$$\cosh(n\xi) = \frac{1}{2} \left( (\cosh(\xi) + \sinh(\xi))^n + (\cosh(\xi) - \sinh(\xi))^n \right).$$

Letting  $\xi = \operatorname{arccosh}(x)$  in this equation and using that  $\cosh(\xi)^2 - \sinh(\xi)^2 = 1$ , we obtain

$$T_n(x) = \frac{1}{2} \left( (x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right),$$

which is the required formula.  $\triangle$

❁ **Exercise C.4** (Recursion relation). *Show that the Chebyshev polynomials satisfy the relation*

$$\forall n \in \{1, 2, \dots\}, \quad T_{n+1} = 2xT_n - T_{n-1}. \quad (\text{C.6})$$

*Solution.* It is sufficient to show the identity for  $x \in [-1, 1]$ , where the formula (C.1) applies. Using well-known trigonometric identities, we have

$$\begin{aligned} \cos((n+1)\theta) &= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta) \\ \cos((n-1)\theta) &= \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta). \end{aligned}$$

Adding both equations and rearranging, we obtain

$$\cos((n+1)\theta) = 2\cos(n\theta)\cos(\theta) - \cos((n-1)\theta).$$

Therefore, using this equation with  $\theta = \arccos(x)$ , we obtain the statement.  $\triangle$

*Remark C.1.* The recursion relation in [Exercise C.4](#) can be employed to show by recursion that  $T_n(x)$  is indeed a polynomial of degree  $n$ .

⚙️ **Exercise C.5.** Since  $T_n: \mathbf{R} \rightarrow \mathbf{R}$  is a polynomial, it may be written in the standard form

$$T_n(x) = \alpha_n^{(n)}x^n + \dots + \alpha_1^{(n)}x + \alpha_0^{(n)}.$$

Prove that  $\alpha_n^{(n)} = 2^{(n-1)}$  provided that  $n \geq 1$ .

*Solution.* From the definition (C.1), the Chebyshev polynomials of degrees 0 and 1 are given by  $T_0(x) = 1$  and  $T_1(x) = x$ . The statement then follows by recursion, using [Exercise C.4](#).  $\triangle$

⚙️ **Exercise C.6.** Let  $\xi \in \mathbf{R} \setminus (-1, 1)$ . Show that, among all the polynomials in  $\mathbf{P}(n)$  that are bounded from above by 1 in absolute value uniformly over the interval  $(-1, 1)$ , the Chebyshev polynomial  $T_n$  achieves the largest absolute value when evaluated at  $\xi$ .

*Solution.* Reasoning by contradiction, we assume that there exists  $p \in \mathbf{P}(n)$  that satisfies

$$\sup_{x \in (-1, 1)} |p(x)| \leq 1 \quad \text{and} \quad |p(\xi)| > |T_n(\xi)|.$$

Let  $q(x) = p(x)T_n(\xi)/p(\xi)$ . Then by construction  $q(\xi) = T_n(\xi)$  and

$$\sup_{x \in (-1, 1)} |q(x)| < 1.$$

Consequently, denoting by  $x_k$  the points defined in (C.3), we have that

$$\forall k \in \{0, \dots, n\}, \quad (-1)^k (T_n - q)(x_k) > 0.$$

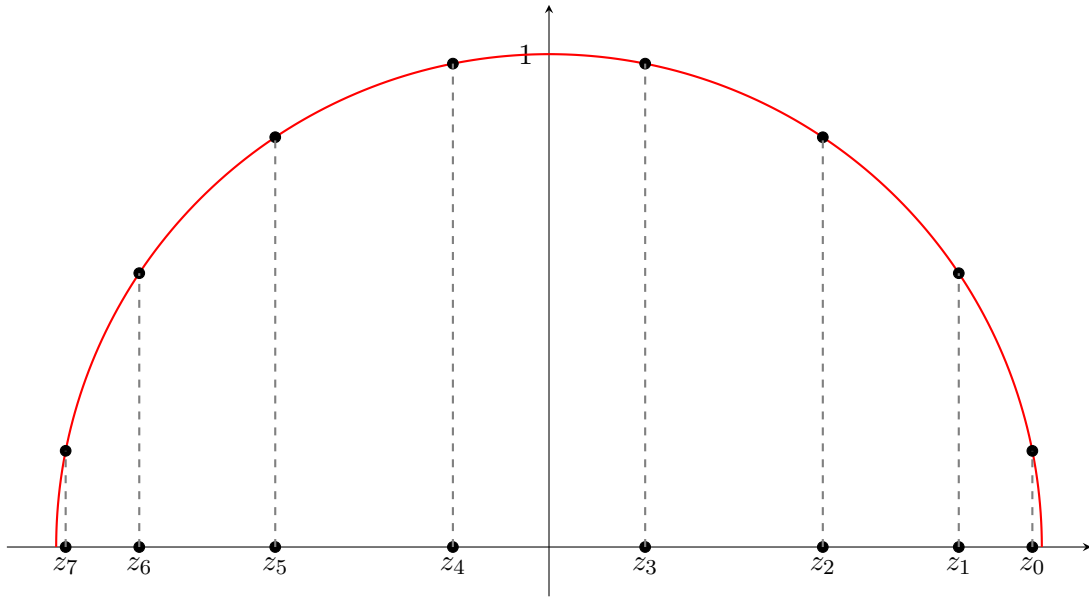
In other words, the polynomial  $T_n - q$  takes positive values at  $\{x_0, x_2, x_4, \dots\}$  and negative values at  $\{x_1, x_3, x_5, \dots\}$ . Consequently, by the intermediate value theorem,  $T_n - q$  possesses  $n$  distinct roots in the open interval  $(-1, 1)$ . Since, in addition,  $(T_n - q)(\xi) = 0$ , we deduce that  $T_n - q$  has  $n + 1$  distinct roots, which is a contradiction given that  $T_n - q$  is a nonzero polynomial of degree at most  $n$ .  $\triangle$

⚙️ **Exercise C.7.** Assume that  $0 < \lambda_1 < \lambda_2$ . Prove that for any polynomial  $p \in \mathbf{P}(n)$  that satisfies  $p(0) = 1$ , it holds that

$$\sup_{\lambda \in (\lambda_1, \lambda_2)} |p(\lambda)| \geq \frac{1}{T_n(\xi)}, \quad \xi := \frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1},$$

with equality for

$$p_*(\lambda) = \frac{T_n\left(\frac{\lambda_1 + \lambda_2 - 2\lambda}{\lambda_2 - \lambda_1}\right)}{T_n\left(\frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1}\right)}. \quad (\text{C.7})$$


 Figure C.2: Roots of the Chebyshev polynomial  $T_8$ .

*Solution.* Assume that  $p \in \mathbf{P}(n)$  is such that  $p(0) = 1$ , and let  $q \in \mathbf{P}(n)$  be given by

$$q(\mu) = p\left(\frac{\lambda_1 + \lambda_2 - (\lambda_2 - \lambda_1)\mu}{2}\right) \quad \Leftrightarrow \quad p(\lambda) = q\left(\frac{\lambda_1 + \lambda_2 - 2\lambda}{\lambda_2 - \lambda_1}\right).$$

Since  $\xi > 1$ , it holds from (C.5) that  $T_n(\xi) > 0$  and it follows from Exercise C.6 that

$$p(0) = q(\xi) \leq T_n(\xi) \sup_{\mu \in (-1,1)} |q(\mu)| = T_n(\xi) \sup_{\lambda \in (\lambda_1, \lambda_2)} |p(\lambda)|,$$

with equality when  $q \propto T_n$ , i.e. when

$$p(\lambda) \propto T_n\left(\frac{\lambda_1 + \lambda_2 - 2\lambda}{\lambda_2 - \lambda_1}\right).$$

The expression (C.7) then follows from the fact that  $p_*(0) = 1$ . △