

Appendix C

Chebyshev polynomials

The Chebyshev polynomials $(T_n)_{n \in \mathbf{N}}$ are given on $[-1, 1]$ by the formula

$$\forall x \in [-1, 1], \quad T_n(x) = \cos(n \arccos(x)). \quad (\text{C.1})$$

Although this formula makes sense only if $x \in [-1, 1]$, the polynomials are defined for all $x \in \mathbf{R}$. Equivalently, the Chebyshev polynomials can be defined from

$$\forall x \in [1, \infty), \quad T_n(x) = \cosh(n \operatorname{arccosh}(x)). \quad (\text{C.2})$$

where $\cosh(\theta) = \frac{1}{2}(e^\theta + e^{-\theta})$ and $\operatorname{arccosh}: [1, \infty) \rightarrow [0, \infty)$ is the inverse function of \cosh . The first few Chebyshev polynomials are illustrated in [Figure C.1](#). It is immediate to show from (C.1) that

- the roots of T_n are given by

$$z_k = \cos\left(\frac{\pi}{2n} + \frac{k\pi}{n}\right), \quad k = 0, \dots, n-1.$$

These are illustrated in [Figure C.2](#).

- The polynomial T_n takes the value 1 or -1 when evaluated at

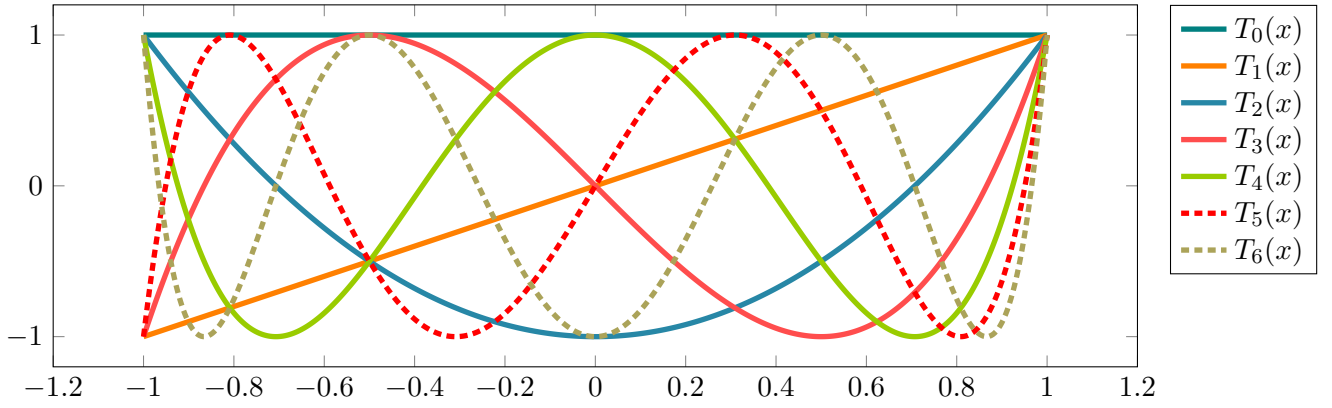
$$x_k = \cos\left(\frac{k\pi}{n}\right), \quad k = 0, \dots, n. \quad (\text{C.3})$$

More precisely, it holds that $T_n(x_k) = (-1)^k$.

⚙️ **Exercise C.1.** Show that (C.1) defines a polynomial of degree n , and find its expression in the usual polynomial notation.

Solution. The key idea is to rewrite the cosine function in terms of the complex exponential:

$$\cos(n\theta) = \frac{1}{2}(e^{in\theta} + e^{-in\theta}) = \frac{1}{2}\left((\cos(\theta) + i\sin(\theta))^n + (\cos(\theta) - i\sin(\theta))^n\right).$$


 Figure C.1: Illustration of the first few Chebyshev polynomials over the interval $[-1, 1]$.

By expanding the power on the right-hand side, we obtain

$$\begin{aligned} (\cos(\theta) + i \sin(\theta))^n &= \sum_{j=0}^n \binom{n}{j} \cos(\theta)^{n-j} i^j \sin(\theta)^j \\ (\cos(\theta) - i \sin(\theta))^n &= \sum_{j=0}^n \binom{n}{j} \cos(\theta)^{n-j} (-i)^j \sin(\theta)^j. \end{aligned}$$

The terms corresponding to odd values of j cancel out in the expression of $\cos(n\theta)$, and so we obtain the following expression for $\cos(n\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$:

$$\begin{aligned} \cos(n\theta) &= \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \cos(\theta)^{n-2j} i^{2j} \sin(\theta)^{2j} \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{2j} \cos(\theta)^{n-2j} (1 - \cos(\theta)^2)^j. \end{aligned}$$

Therefore, we conclude that

$$T_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} x^{n-2j} (x^2 - 1)^j. \quad (\text{C.4})$$

△

⚙️ **Exercise C.2.** Show that the same polynomials are obtained from (C.2).

Solution. Notice that

$$\begin{aligned} \cosh(n\xi) &= \frac{1}{2} (e^{n\xi} + e^{-n\xi}) \\ &= \frac{1}{2} \left((\cosh(\xi) + \sinh(\xi))^n + (\cosh(\xi) - \sinh(\xi))^n \right). \end{aligned}$$

Using the binomial formula, we obtain

$$\begin{aligned} \cosh(n\xi) &= \frac{1}{2} \sum_{j=0}^n \binom{n}{j} (\cosh(\xi)^{n-j} \sinh(\xi)^j + \cosh(\xi)^{n-j} (-1)^j \sinh(\xi)^j) \\ &= \frac{1}{2} \sum_{j=0}^n \binom{n}{j} \cosh(\xi)^{n-j} (\sinh(\xi)^j + (-1)^j \sinh(\xi)^j). \end{aligned}$$

The contributions of the odd values of j cancel out, and so we obtain

$$\cosh(n\xi) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \cosh(\xi)^{n-2j} \sinh(\xi)^{2j}.$$

Since $\cosh(\xi)^2 - \sinh(\xi)^2 = 1$, we deduce that

$$\cosh(n\xi) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j} \cosh(\xi)^{n-2j} (\cosh(\xi)^2 - 1)^j,$$

which after the substitution of $\xi = \operatorname{arccosh}(x)$ leads to (C.4). \triangle

✿ **Exercise C.3** (Yet another expression for the Chebyshev polynomials). *Show that $T_n(x)$ may be defined from the formula*

$$T_n(x) = \frac{1}{2} \left(x + \sqrt{x^2 - 1} \right)^n + \frac{1}{2} \left(x - \sqrt{x^2 - 1} \right)^n \quad \text{for } |x| \geq 1. \quad (\text{C.5})$$

Solution. We showed in the solution of Exercise C.2 that

$$\cosh(n\xi) = \frac{1}{2} \left((\cosh(\xi) + \sinh(\xi))^n + (\cosh(\xi) - \sinh(\xi))^n \right).$$

Letting $\xi = \operatorname{arccosh}(x)$ in this equation and using that $\cosh(\xi)^2 - \sinh(\xi)^2 = 1$, we obtain

$$T_n(x) = \frac{1}{2} \left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right),$$

which is the required formula. \triangle

✿ **Exercise C.4** (Recursion relation). *Show that the Chebyshev polynomials satisfy the relation*

$$\forall n \in \{1, 2, \dots\}, \quad T_{n+1} = 2xT_n - T_{n-1}. \quad (\text{C.6})$$

Solution. It is sufficient to show the identity for $x \in [-1, 1]$, where the formula (C.1) applies. Using well-known trigonometric identities, we have

$$\begin{aligned} \cos((n+1)\theta) &= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta) \\ \cos((n-1)\theta) &= \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta). \end{aligned}$$

Adding both equations and rearranging, we obtain

$$\cos((n+1)\theta) = 2\cos(n\theta)\cos(\theta) - \cos((n-1)\theta).$$

Therefore, using this equation with $\theta = \arccos(x)$, we obtain the statement. \triangle

Remark C.1. The recursion relation in Exercise C.4 can be employed to show by recursion that $T_n(x)$ is indeed a polynomial of degree n .

⚙️ **Exercise C.5.** Since $T_n: \mathbf{R} \rightarrow \mathbf{R}$ is a polynomial, it may be written in the standard form

$$T_n(x) = \alpha_n^{(n)}x^n + \dots + \alpha_1^{(n)}x + \alpha_0^{(n)}.$$

Prove that $\alpha_n^{(n)} = 2^{(n-1)}$ provided that $n \geq 1$.

Solution. From the definition (C.1), the Chebyshev polynomials of degrees 0 and 1 are given by $T_0(x) = 1$ and $T_1(x) = x$. The statement then follows by recursion, using Exercise C.4. \triangle

⚙️ **Exercise C.6.** Let $\xi \in \mathbf{R} \setminus (-1, 1)$. Show that, among all the polynomials in $\mathbf{P}(n)$ that are bounded from above by 1 in absolute value uniformly over the interval $(-1, 1)$, the Chebyshev polynomial T_n achieves the largest absolute value when evaluated at ξ .

Solution. Reasoning by contradiction, we assume that there exists $p \in \mathbf{P}(n)$ that satisfies

$$\sup_{x \in (-1, 1)} |p(x)| \leq 1$$

and $|p(\xi)| > |T_n(\xi)|$. Let $q(x) = p(x)T_n(\xi)/p(\xi)$. Then by construction $q(\xi) = T_n(\xi)$ and

$$\sup_{x \in (-1, 1)} |q(x)| < 1.$$

Consequently, denoting by x_k the points defined in (C.3), we have that

$$\forall k \in \{0, \dots, n\}, \quad (-1)^k (T_n - q)(x_k) > 0.$$

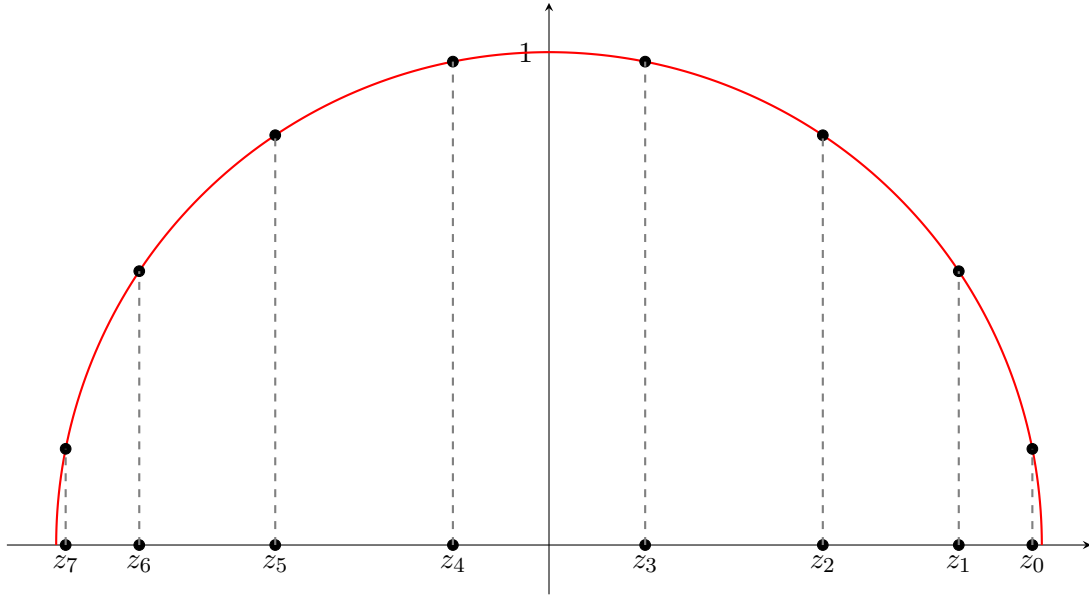
In other words, the polynomial $T_n - q$ takes positive values at $\{x_0, x_2, x_4, \dots\}$ and negative values at $\{x_1, x_3, x_5, \dots\}$. Consequently, by the intermediate value theorem, $T_n - q$ possesses n distinct roots in the open interval $(-1, 1)$. Since, in addition, $(T_n - q)(\xi) = 0$, we deduce that $T_n - q$ has $n + 1$ distinct roots, which is a contradiction given that $T_n - q$ is a nonzero polynomial of degree at most n . \triangle

⚙️ **Exercise C.7.** Assume that $0 < x_1 < x_2$. Prove that for any polynomial $p \in \mathbf{P}(n)$ that satisfies $p(0) = 1$, it holds that

$$\sup_{\lambda \in (\lambda_1, \lambda_2)} |p(\lambda)| \geq \frac{1}{T_n(\xi)}, \quad \xi := \frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1},$$

with equality for

$$p_*(\lambda) = \frac{T_n\left(\frac{\lambda_1 + \lambda_2 - 2\lambda}{\lambda_2 - \lambda_1}\right)}{T_n\left(\frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1}\right)}.$$


 Figure C.2: Roots of the Chebyshev polynomial T_8 .

Solution. Assume that $p \in \mathbf{P}(n)$ is such that $p(0) = 1$, and let $q \in \mathbf{P}(n)$ be given by

$$q(\mu) = p\left(\frac{\lambda_1 + \lambda_2 - (\lambda_2 - \lambda_1)\mu}{2}\right) \quad \Leftrightarrow \quad p(\lambda) = q\left(\frac{\lambda_1 + \lambda_2 - 2\lambda}{\lambda_2 - \lambda_1}\right).$$

Since $\xi > 1$, it follows from [Exercise C.6](#) that

$$p(0) = q(\xi) \leq T_n(\xi) \sup_{\mu \in (-1,1)} |q(\mu)| = T_n(\xi) \sup_{\lambda \in (\lambda_1, \lambda_2)} |p(\lambda)|,$$

with equality when $q \propto T_n$, i.e. when

$$p(\lambda) \propto T_n\left(\frac{\lambda_1 + \lambda_2 - 2\lambda}{\lambda_2 - \lambda_1}\right).$$

The statement then follows from the fact that $p(0) = 1$. △

Solution. [Exercise C.7](#) may alternatively be solved without a change of variable, by using a more direct approach similar to that in [Exercise C.6](#). Reasoning by contradiction, we assume that there is $q \in \mathbf{P}(n)$ such that

$$\sup_{\lambda \in (\lambda_1, \lambda_2)} |q(\lambda)| < T_n\left(\frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1}\right)^{-1}. \quad (\text{C.7})$$

Then $p_* - q$ is a polynomial of degree n with a zero at $\lambda = 0$. For $k \in \{0, \dots, n\}$, let

$$x_k = \frac{\lambda_2 + \lambda_1 - (\lambda_2 - \lambda_1) \cos\left(\frac{k\pi}{n}\right)}{2}.$$

By (C.3), it holds that

$$p_*(x_k) = \frac{T_n\left(\cos\left(\frac{k\pi}{n}\right)\right)}{T_n\left(\frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1}\right)} = \frac{(-1)^j}{T_n\left(\frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1}\right)}$$

From this equation and (C.7), it follows that

$$\forall k \in \{0, \dots, n\}, \quad (p_* - q)(x_k) \text{ is } \begin{cases} \text{strictly positive if } k \text{ is even,} \\ \text{strictly negative if } k \text{ is odd.} \end{cases}$$

This implies, by the intermediate value theorem, that $p_* - q$ has n roots in the interval $[\lambda_1, \lambda_2]$, so $n + 1$ roots in total, but this is impossible for a nonzero polynomial of degree n . \triangle