Appendix C

Chebyshev polynomials

The Chebyshev polynomials $(T_n)_{n\in\mathbf{N}}$ are given on $[-1,1]$ by the formula

$$\forall x \in [-1,1], \qquad T_n(x) = \cos\big(n \arccos(x)\big). \tag{C.1}$$

Although this formula makes sense only if $x \in [-1,1]$, the polynomials are defined for all $x \in \mathbf{R}$. Equivalently, the Chebyshev polynomials can be defined from

$$\forall x \in [1,\infty), \qquad T_n(x) = \cosh\big(n \operatorname{arccosh}(x)\big). \tag{C.2}$$

where $\cosh(\theta) = \frac{1}{2}\left(\mathrm{e}^{\theta} + \mathrm{e}^{-\theta}\right)$ and $\operatorname{arccosh} = \cosh^{-1}(\theta)$ is the inverse function, which is uniquely defined for the codomain $[0,\infty)$. The first few Chebyshev polynomials are illustrated in Figure C.1.

Exercise C.1. *Show that (C.1) defines a polynomial of degree n, and find its expression in the usual polynomial notation.*

> *Solution.* The key idea is to rewrite the cosine function as the real part of a complex exponential:
>
> $$\cos(n\theta) = \Re\left(\mathrm{e}^{\mathrm{i}n\theta}\right) = \Re\Big(\big(\cos(\theta) + \mathrm{i}\sin(\theta)\big)^n\Big).$$

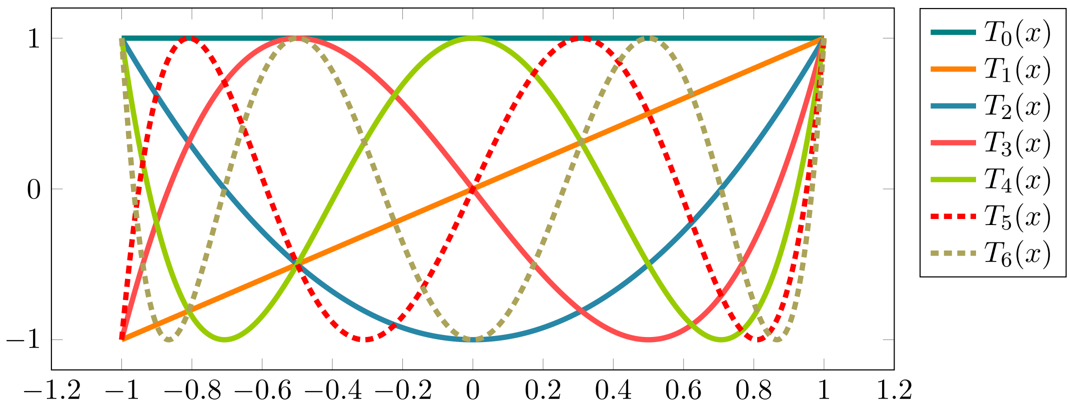

Figure C.1: Illustration of the first few Chebyshev polynomials over the interval $[-1,1]$.

By expanding the power on the right-hand side, we can obtain an expression for $\cos(n\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$. By the binomial formula, it holds that

$$\big(\cos(\theta) + \mathrm{i}\sin(\theta)\big)^n = \sum_{j=0}^{n} \binom{n}{j} \cos(\theta)^{n-j}\, \mathrm{i}^j\, \sin(\theta)^j.$$

The terms corresponding to odd values of j are imaginary, and so these cancel out after taking the real part, which leads to

$$\begin{aligned}\cos(n\theta) &= \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \cos(\theta)^{n-2j} \mathrm{i}^{2j} \sin(\theta)^{2j} \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{2j} \cos(\theta)^{n-2j} \big(1 - \cos(\theta)^2\big)^j.\end{aligned}$$

Therefore, we conclude that

$$T_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} x^{n-2j} \big(x - 1\big)^j. \tag{C.3}$$

△

Exercise C.2. *Show that the same polynomials are obtained from* (C.2).

Solution. Notice that

$$\begin{aligned}\cosh(n\xi) &= \frac{1}{2}\left(\mathrm{e}^{n\xi} + \mathrm{e}^{-n\xi}\right) \\ &= \frac{1}{2}\Big(\big(\cosh(\xi) + \sinh(\xi)\big)^n + \big(\cosh(\xi) - \sinh(\xi)\big)^n\Big).\end{aligned}$$

Using the binomial formula, we obtain

$$\begin{aligned}\cosh(n\xi) &= \frac{1}{2}\sum_{j=0}^{n} \binom{n}{j} \left(\cosh(\xi)^{n-j} \sinh(\xi)^j + \cosh(\xi)^{n-1} (-1)^j \sinh(\xi)^j\right) \\ &= \frac{1}{2}\sum_{j=0}^{n} \binom{n}{j} \cosh(\xi)^{n-j} \left(\sinh(\xi)^j + (-1)^j \sinh(\xi)^j\right).\end{aligned}$$

The contributions of the odd values of j cancel out, and so we obtain

$$\cosh(n\xi) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \cosh(\xi)^{n-2j} \sinh(\xi)^{2j}.$$

Since $\cosh(\xi)^2 - \sinh(\xi)^2 = 1$, we deduce that

$$\cosh(n\xi) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j} \cosh(\xi)^{n-2j} (\cosh(\xi)^2 - 1)^j,$$

which after the substitution of $\xi = \operatorname{arccosh}(x)$ leads to (C.3). △

Exercise C.3 (Recursion relation)**.** *Show that the Chebyshev polynomials satisfy the relation*

$$\forall n \in \{2, 3, \dots\}, \qquad T_{n+1} = 2xT_n - T_{n-1}.$$

Solution. It is sufficient to show the identity for $x \in [-1, 1]$, where the formula (C.1) applies. Using well-known trigonometric identities, we have

$$\cos\big((n+1)\theta\big) = \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta)$$
$$\cos\big((n-1)\theta\big) = \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta).$$

Adding both equations and rearranging, we obtain

$$\cos\big((n+1)\theta\big) = 2\cos(n\theta)\cos(\theta) - \cos\big((n-1)\theta\big).$$

Therefore, using this equation with $\theta = \arccos(x)$, we obtain the statement. △

Remark C.1. The recursion relation in Exercise C.3 can be employed to show by recursion that $T_n(x)$ is indeed a polynomial of degree n.

Exercise C.4. *Since $T_n \colon \mathbf{R} \to \mathbf{R}$ is a polynomial, it may be written in the standard form*

$$T_n(x) = \alpha_n^{(n)} x^n + \dots + \alpha_1^{(n)} x + \alpha_0^{(n)}.$$

Prove that $\alpha_n^{(n)} = 2^{(n-1)}$ provided that $n \geq 1$.

Solution. From the definition (C.1), the Chebyshev polynomials of degrees 0 and 1 are given by $T_0(x) = 1$ and $T_1(x) = x$. The statement then follows by recursion, using Exercise C.3. △

It is immediate to show from (C.1) that the roots of T_n are given by

$$z_k = \cos\left(\frac{\pi}{2n} + \frac{k\pi}{n}\right), \qquad k = 0, \dots, n-1.$$

The roots are illustrated in Figure C.2. The polynomial T_n takes the value 1 or -1 when evaluated at

$$x_k = \cos\left(\frac{k\pi}{n}\right), \qquad k = 0, \dots, n. \tag{C.4}$$

More precisely, it holds that $T_n(x_k) = (-1)^k$.

Exercise C.5. *Let $\xi \in \mathbf{R} \backslash (-1, 1)$. Show that, among all the polynomials in $\mathbf{P}(n)$ that are bounded from above by 1 in absolute value uniformly over the interval $(-1, 1)$, the Chebyshev polynomial T_n achieves the largest absolute value when evaluated at ξ.*

Solution. Reasoning by contradiction, we assume that p satisfies

$$\sup_{x \in (-1,1)} |p(x)| \leq 1$$

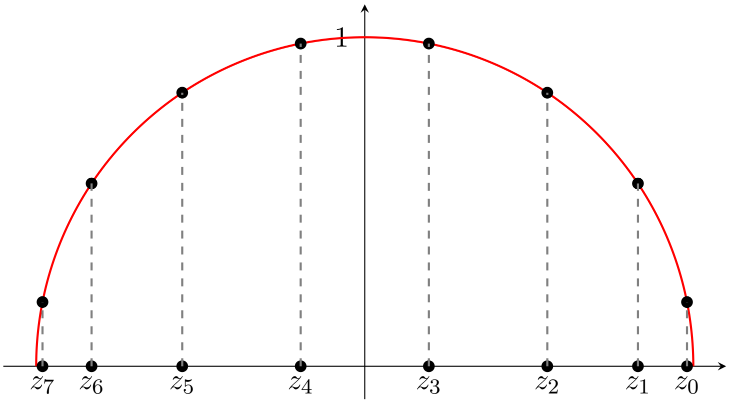

Figure C.2: Roots of the Chebyshev polynomial T_8.

and $|p(\xi)| > |T_n(\xi)|$. Let $q(x) = p(x)T_n(\xi)/p(\xi)$. Then by construction $q(\xi) = T_n(\xi)$ and

$$\sup_{x\in(-1,1)} |q(x)| < 1.$$

Consequently, denoting by x_k the points defined in (C.4), we have that

$$\forall k \in \{0, \dots, n\}, \qquad (-1)^k (T_n - q)(x_k) > 0.$$

In other words, the polynomial $T_n - q$ takes positive values at $\{x_0, x_2, x_4, \dots\}$ and negative values at $\{x_1, x_3, x_5, \dots\}$. Consequently, by the intermediate value theorem, $T_n - q$ possesses n distinct roots in the open interval $(-1, 1)$. Since, in addition, $(T_n - q)(\xi) = 0$, we deduce that $T_n - q$ has $n + 1$ distinct roots, which is a contradiction given that $T_n - q$ is a nonzero polynomial of degree at most n. △